Reduction theory for symmetry breaking with applications to nematic systems

François Gay-Balmaz[1] and Cesare Tronci[2]

[1] *Control and Dynamical Systems, California Institute of Technology*

[2] *Section de Mathématiques, École Polytechnique Fédérale de Lausanne. Switzerland*

Abstract

We formulate Euler-Poincaré and Lagrange-Poincaré equations for systems with broken symmetry. We specialize the general theory to present explicit equations of motion for nematic systems, ranging from single nematic molecules to biaxial liquid crystals. The geometric construction applies to order parameter spaces consisting of either unsigned unit vectors (directors) or symmetric matrices (alignment tensors). On the Hamiltonian side, we provide the corresponding Poisson brackets in both Lie-Poisson and Hamilton-Poincaré formulations. The explicit form of the helicity invariant for uniaxial nematics is also presented, together with a whole class of invariant quantities (Casimirs) for two dimensional incompressible flows.

1 Introduction

1.1 A simple example: the heavy top

Symmetry breaking phenomena are widely common in several physical contexts, from classical mechanics to particle physics. The simplest example is probably the heavy top dynamics, that is the motion of a rigid body with a fixed point in a gravitational field. This motion takes place on the special Euclidean group $SE(3)$, which carries a semidirect product structure $SE(3) = SO(3) \circledS \mathbb{R}^3$ involving the special orthogonal group $SO(3)$. Semidirect product Lie group structures of this kind may be understood as "signatures" of broken symmetries in physical systems. For the case of the heavy top, the physical description involves a basic configuration space which is $SO(3)$, i.e. the same as in rigid body dynamics. Although one can simply write Hamilton's equations on the cotangent bundle $T^*SO(3)$, one realizes that the heavy top dynamics is very different from the simple rigid body case. Indeed, the presence of gravity in the system produces a Hamiltonian which is *not* $SO(3)$-invariant, contrarily to what happens for the rigid body. Thus, besides the body angular momentum, one also considers the direction of gravity (in the body reference frame) as a dynamical variable in S^2. Upon identifying S^2 with unit vectors in $\mathbb{R}^3$, one is leaded to enlarge the Lie group $SO(3)$ thereby considering the semidirect product $SO(3) \circledS \mathbb{R}^3$. Such an approach has a relevant infinite-dimensional analogue which yields the theory of compressible fluid flows. The main references for geometric mechanics on semidirect product Lie groups are [32, 22]. One

of the targets of the present paper is to generalize these semidirect product structures in the context of reduction by symmetry, thereby extending to the case when the vector space V in the semidirect product $G \circledS V$ is replaced by a manifold M, so that the semidirect-product group structure breaks into the Cartesian product $G \times M$. This introduction reviews the high points of the Euler-Poincaré reduction theory and summarizes the results obtained in the paper, after a brief discussion of the geometric methods that will be used.

1.2 Euler-Poincaré approach to heavy tops

This section briefly reviews the concrete mathematical background for the dynamics of the heavy top. This example will be still considered in the remainder of the paper for direct comparison.

Although it is defined on the configuration space $SO(3)$, the heavy top dynamics exhibits rotational symmetry breaking through the appearance of a potential term, depending on the direction of gravity $\mathbf{e}_3 \in S^2$. The heavy top Lagrangian $L_{\mathbf{e}_3} : TSO(3) \to \mathbb{R}$ is written as

$$L_{\mathbf{e}_3}(\chi, \dot{\chi}) = \frac{1}{2} \int_{\mathcal{B}} \rho(A) |\dot{\chi} A|^2 d^3A - Mg\ell\, \mathbf{e}_3 \cdot \chi\boldsymbol{\zeta} \tag{1.1}$$

where $\mathcal{B} \subset \mathbb{R}^3$ is the reference configuration of the body, $\rho(A)$ is the mass density, M is the total mass, g is the acceleration of gravity, and $\boldsymbol{\zeta}$ is the unit vector along the segment of length ℓ that connects the fixed point of the body with its center of mass at $t = 0$. Evidently, the $SO(3)$-symmetry is broken: $L_{\mathbf{e}_3}(\chi, \dot{\chi}) \neq L_{\mathbf{e}_3}(\chi^{-1}\dot{\chi})$. Rather, the system is invariant only with respect to $SO(2)$ (rotations around the vertical axis), since this is the isotropy subgroup of $\mathbf{e}_3$. At this point, one denotes $\mathbf{e}_3 = \boldsymbol{\Gamma}_0 \in \mathbb{R}^3$ and defines $L : TSO(3) \times \mathbb{R}^3 \to \mathbb{R}$ by $L(\chi, \dot{\chi}, \boldsymbol{\Gamma}_0) := L_{\boldsymbol{\Gamma}_0}(\chi, \dot{\chi})$. where $\boldsymbol{\Gamma}_0$ is interpreted as a new variable. We now let $SO(3)$ act on the extended space $TSO(3) \times \mathbb{R}^3$ by the left action $(\chi, \dot{\chi}, \boldsymbol{\Gamma}_0) \mapsto (\psi\chi, \psi\dot{\chi}, \psi\boldsymbol{\Gamma}_0)$, with the result that the function L is now $SO(3)$-invariant since

$$Mg\ell\, \psi\boldsymbol{\Gamma}_0 \cdot \psi\chi\boldsymbol{\zeta} = Mg\ell\, \boldsymbol{\Gamma}_0 \cdot \chi\boldsymbol{\zeta}, \quad \text{for all } \psi \in SO(3).$$

Thus, by $SO(3)$-invariance, L induces the reduced Lagrangian $l : \mathfrak{so}(3) \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$l(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = \frac{1}{2} \mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} - Mg\ell \boldsymbol{\Gamma} \cdot \boldsymbol{\zeta}.$$

where $\hat{\boldsymbol{\Omega}} = \chi^{-1}\dot{\chi}$ (so that 'hat' denotes the usual isomorphism $\mathfrak{so}(3) \simeq \mathbb{R}^3$) and $\boldsymbol{\Gamma} = \chi^{-1}\boldsymbol{\Gamma}_0$. The moment of inertia tensor $\mathbb{I}$ is defined by (cf. [30])

$$\mathbb{I} = -\int_{\mathcal{B}} \rho(A) \left(AA^T - |A|^2 \mathbf{I} \right) d^3A\,.$$

Again, the reduced Lagrangian produces the Euler-Poincaré variational principle

$$\delta \int_{t_0}^{t_1} l(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) dt = 0$$

thereby yielding the equations

$$\frac{d}{dt}\frac{\delta l}{\delta \boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \frac{\delta l}{\delta \boldsymbol{\Omega}} = -\boldsymbol{\Gamma} \times \frac{\delta l}{\delta \boldsymbol{\Gamma}} \qquad \frac{d\boldsymbol{\Gamma}}{dt} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}$$

where the variations are evaluated as $\delta\mathbf{\Omega} = \delta(\chi^{-1}\dot{\chi}) = \dot{\mathbf{\Sigma}} + \mathbf{\Omega} \times \mathbf{\Sigma}$ (with $\mathbf{\Sigma} := \chi^{-1}\delta\chi$) and $\delta\mathbf{\Gamma} = \mathbf{\Omega} \times \mathbf{\Gamma}$. This construction is referred to as Euler-Poincaré approach for semidirect products (cf. [22]). In this particular case, the configuration Lie group is the well known special Euclidean group $SE(3) := SO(3) \circledS \mathbb{R}^3$. Such a formulation is only possible upon identifying points on the sphere (e.g. the direction of gravity) with unit vectors in $\mathbb{R}^3$ and by noticing that the modulus of the latter is preserved by the motion. Although in many situations this process leads to no contradiction, this is not always the case in condensed matter physics and it is useful to consider parameters (called 'order parameters') belonging to a manifold M rather than to a vector space V. For example, the heavy top would require a Lagrangian $L : TSO(3) \times S^2 \to \mathbb{R}$. The next section gives an overview of the general setting in the context of condensed matter physics.

1.3 Broken symmetries in condensed matter: nematic molecules

An enlightening example of broken symmetry appears in the theory of nematic liquid crystals [6, 8]. In the simplest case of uniaxial molecules, such systems are continuum systems composed of oriented particles, i.e. particles that are endowed with a special orientation identified with an unsigned unit vector $\mathbf{n} \sim -\mathbf{n}$, called *director*. The presence of such a special direction in the system plays exactly the same role as in heavy top dynamics, as it is shown by the equations of motion for the single nematic particle:

$$\frac{d}{dt}\frac{\delta l}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \frac{\delta l}{\delta \boldsymbol{\nu}} + \mathbf{n} \times \frac{\delta l}{\delta \mathbf{n}}\,, \qquad \dot{\mathbf{n}} = \boldsymbol{\nu} \times \mathbf{n}\,. \tag{1.2}$$

A direct comparison with heavy top dynamics shows the strict similarity between nematic particles and heavy tops. However, a director carries no sign and thus it does not belong to S^2, but rather it takes values in $S^2/\mathbb{Z}_2$, i.e. the unit sphere S^2 with diametrically opposite points identified, also known as the real projective plane $\mathbb{R}P^2$. Therefore the occurrence of the director breaks the full $SO(3)$ rotational symmetry of the particle, whose rotational motion is now invariant only under a subgroup $\mathcal{P} \subset SO(3)$, i.e. the isotropy subgroup of the director itself. For nematic particles, the isotropy subgroup of unsigned unit vectors is $\mathcal{P} = O(2)$. (More precisely $\mathcal{P} = D_\infty$, where the group D_∞ consists of rotations about the molecular axis and 180°-rotations about a normal to the molecular axis). In more general condensed matter applications, if $\mathcal{O}$ is the broken symmetry (the order parameter group) acting *transitively* on the order parameter space M, then the latter is isomorphic to the coset space $\mathcal{O}/\mathcal{P}$, where $\mathcal{P}$ is the isotropy subgroup $\mathcal{P} = \mathcal{O}_{n_0}$ of a reference point $n_0 \in M$. In the case of nematics, the order parameter space is $\mathcal{O}/\mathcal{P} = SO(3)/O(2) \simeq \mathbb{R}P^2$. Thus, after starting with the configuration space $\mathcal{O}$ ($SO(3)$ for nematics), the symmetry breaking requires to consider also dynamics on the order parameter space $M \simeq \mathcal{O}/\mathcal{P}$. Notice that, this description refers only to the rotational motion of complex fluid particles and we have forgotten about its motion in physical space. However, the latter can be included a posteriori by considering Cartesian product configuration manifolds such as $Q \times \mathcal{O}$, where Q denotes physical space and we assume that $\mathcal{O}$ does not act on Q. In the first part of this paper we shall neglect motion in physical space in order to keep the treatment sufficiently compact.

Nematic particles differ from the heavy top in that the order parameter space can be a manifold, rather than a vector space. Thus, the ordinary theory of heavy top dynamics needs to be extended for condensed matter applications. In particular, even if the order parameter coset space $\mathcal{O}/\mathcal{P}$ is naturally associated to symmetry breaking, it is not clear how this coset space

arises from a fundamental approach in terms of reduction by symmetry. For example, the emergence of a broken symmetry often leads to consider $\mathcal{O}/\mathcal{P}$ (rather than $\mathcal{O}$) as the configuration manifold. In the context of nematic particles, the Ericksen-Leslie equations (cf. [27]) are simply the Euler-Lagrange equations on the projective plane $\mathbb{R}P^2 \simeq SO(3)/O(2)$. One of the tasks of this work is to show how this new configuration space emerges naturally from reduction theory. In more generality,

this paper aims to provide a unified systematic framework for systems with broken symmetry by presenting the corresponding Euler-Poincaré and Lagrange-Poincaré approaches to formulate their dynamics.

We shall mainly concentrate on the symmetry properties of finite dimensional systems, while the last part of the paper extends the treatment to consider the infinite dimensional cases. In terms of complex fluid systems, this means that we shall focus on the symmetry properties of the fluid single-particle dynamics. The question of how the underlying geometric structure is preserved in passing from the microscopic single-particle approach to the macroscopic continuum description will be the subject of future work.

Notation. We have used the notation $\mathcal{O}$ for the order parameter group of a system with broken symmetry and have denoted by M the order parameter space, on which $\mathcal{O}$ act transitively. From now on we will use the notation $\mathcal{O}$ only when the group acts transitively on M. For general actions, the Lie group will be denoted by G.

1.4 Geometric setting for symmetry breaking

This section introduces some of the mathematical theory of systems with broken symmetry and anticipates how the coset space $\mathcal{O}/\mathcal{P}$ emerges in a more rigorous framework.

Let $\mathcal{O}$ be a Lie group (the order parameter group), acting *transitively* on the *left* of a manifold M (the order parameter space). Choose an element $n_0 \in M$, and consider the isotropy subgroup $\mathcal{P} := \mathcal{O}_{n_0}$. We have the isomorphism $\mathcal{O}/\mathcal{P} \to M, \quad [\chi] = \chi\,\mathcal{P} \mapsto \chi n_0$, where $\mathcal{P}$ acts on $\mathcal{O}$ by right multiplication. The dynamics is described by a $\mathcal{P}$-invariant Lagrangian $L_{n_0} : T\mathcal{O} \to \mathbb{R}$, which produces the Euler-Lagrange equations on $T\mathcal{O}$. Now we notice that, from the transitivity of the action, any $\mathcal{P}$-invariant function L_{n_0} determines a unique function

$$L : T\mathcal{O} \times M \to \mathbb{R}, \quad L(v_\chi, \varphi n_0) := L_{n_0}(v_\chi \varphi)\,, \quad \text{with} \quad \varphi \in \mathcal{O}$$

that is $\mathcal{O}$-invariant under the right action of $\psi \in \mathcal{O}$ given by $(v_\chi, n) \mapsto (v_\chi \psi, \psi^{-1} n)$. When L_{n_0} is hyperregular, one can obtain the Hamiltonian formulation associated to a $\mathcal{P}$-invariant Hamiltonian $H_{n_0} : T^*\mathcal{O} \to \mathbb{R}$ by Legendre transformation. By proceeding analogously, one defines an $\mathcal{O}$-invariant Hamiltonian $H(\alpha_\chi, \varphi n_0) = H_{n_0}(\alpha_\chi \varphi)$ on $T^*\mathcal{O} \times M$ (with the obvious notation $\alpha_\chi \in T^*\mathcal{O}$).

As explained in the previous section, this paper uses symmetry reduction theory to present the equations of motion on the corresponding reduced space. In particular, given a arbitrary *left* action $n \mapsto gn$ of a Lie group G on a manifold M and a G-invariant function $L : TG \times M \to \mathbb{R}$ under the *right* action $(v_h, n) \mapsto (v_h g, g^{-1} n)$,

we develop two approaches:

(1) **Euler-Poincaré approach to symmetry breaking** (Section 2). We make use of the diffeomorphism

$$(TG \times M)/G \to \mathfrak{g} \times M, \quad [v_g, n] \mapsto \left(v_g g^{-1}, gn\right),$$

so that the function L induces a reduced Lagrangian $l : \mathfrak{g} \times M \to \mathbb{R}$, which is defined by $l(v_g g^{-1}, gn) = L(v_g, n)$. We obtain the equation of motion on the reduced space $\mathfrak{g} \times M$. This construction generalizes the well known Euler-Poincaré theory for semidirect products. We explore the Hamiltonian side and determine the associated noncanonical Poisson brackets. In this context, we obtain a restriction of the Lie-Poisson bracket of [26], who considered the case when the manifold M is Poisson. Indeed, the Legendre transform of the Euler-Poincaré equations yields the bracket in [26] in the special case when M is endowed with the trivial Poisson structure. As from the discussion above, a particularly interesting situation is the case of a transitive action of $G = \mathcal{O}$ on M, so that M is isomorphic to the coset space $\mathcal{O}/\mathcal{P}$, $\mathcal{P}$ being the isotropy subgroup of $\mathcal{O}$ for a fixed element $n_0 \in M$. This result justifies the usual emergence of coset spaces for symmetry breaking in terms of Euler-Poincaré reduction.

(2) **Lagrange-Poincaré approach to symmetry breaking** (Section 4). We shall derive Lagrange-Poincaré equations by applying standard Lagrangian reduction to the ordinary Lagrangian $L_{n_0} : TG \to \mathbb{R}$, defined by $L_{n_0}(v_g) := L(v_g, n_0)$ for a *fixed reference element* $n_0 \in M$. Since L_{n_0} is invariant under the isotropy group G_{n_0} of n_0, this process involves the quotient TG/G_{n_0}. For simplicity, we suppose here that $G = \mathcal{O}$ acts transitively on M and denote by $\mathcal{P}$ the isotropy group of n_0 (the case of a general action is treated in Section 4.2). In order to explicitly write the reduced equations, one needs a connection on the right principal bundle $\mathcal{O} \to \mathcal{O}/\mathcal{P} \simeq M$. Using this connection, the reduced tangent bundle $T\mathcal{O}/\mathcal{P}$ can be identified with the vector bundle $TM \oplus_M \widetilde{\mathfrak{p}}$ over $M \simeq \mathcal{O}/\mathcal{P}$, where $\oplus$ denotes Whitney sum, $\widetilde{\mathfrak{p}}$ denotes the adjoint bundle $(\mathcal{O} \times \mathfrak{p})/\mathcal{P}$ and gothic fonts denote Lie algebras of corresponding Lie groups, as usual. This is a general construction [5], which is here applied to the coset bundle $\mathcal{O} \to \mathcal{O}/\mathcal{P}$.

1.5 Summary of main results

After slightly extending the ordinary Euler-Poincaré theory to account for order parameter manifolds, this construction is applied to recover the equations of motion for nematic particles, thereafter formulating new Euler-Poincaré equations for biaxial nematic particles and V-shaped molecules. As a further step, the Euler-Poincaré equations for ordinary nematic and biaxial particles are also expressed in terms of the corresponding alignment tensors, e.g. $\mathsf{Q} = 1/2\left(\mathbf{n}\mathbf{n}^T - 1/3\,\mathbf{I}\right)$ for nematics.

The second important result concerns the application of the Lagrange-Poincaré reduction method to the coset bundle $G \to G/G_{n_0}$. Indeed, this method is shown to be extremely powerful when combined with the definition of a mechanical connection on $G \to G/G_0$. The result of this combination is the formulation of the nematic particle motion on the configuration space $M = SO(3)/O(2)$, that allows to relate systematically the variables $(\mathbf{n}, \dot{\mathbf{n}}) \in TM$ with the corresponding Euler-Poincaré variables $(\boldsymbol{\nu}, \mathbf{n}) \in \mathfrak{so}(3) \times M$. This step requires particular care because of the appearance of the extra constant $r = \mathbf{n} \cdot \boldsymbol{\nu}$, belonging to the commutative Lie algebra $\mathfrak{o}(2)$. For example, setting $r = 0$ neglects rotations about $\mathbf{n}$, consistently with the rod-like nature of nematic particles, while this is not sensible for heavy top dynamics, which turns

out to be also explained by exactly the same procedure (upon replacing $O(2)$ with $SO(2)$). It is worth mentioning that the possibility of a non-zero quantity $r = \mathbf{n} \cdot \boldsymbol{\nu}$ for the heavy top does not represent a limit of the theory: rather it represents a strong point of this approach that enables to split Euler-Lagrange (EL) dynamics on $T(G/G_{n_0}) = TS^2$ from Euler-Poincaré dynamics on the Lie algebra $\mathfrak{g}_{n_0} = \mathfrak{so}(2)$ (i.e. trivial motion, in this specific case). As a final result of this method, we obtain the Ericksen-Leslie equation as a covariant EL equation on TS^2 and we extend its validity to the heavy top case, including a non-zero $r \in \mathfrak{so}(2)$.

In the last part of this paper, the finite-dimensional treatment is extended to the infinite-dimensional fluid theory of liquid crystals. This is performed in two reduction stages: the first corresponding to the broken symmetry in the micromotion of nematic particles and the second corresponding to the fluid relabeling symmetry. On the Lagrange-Poincaré side, this requires the use of a recent reduction procedure (known as 'metamorphosis'), first appeared in the study of the shape evolution in image dynamics [24]. After establishing the direct correspondence between the dynamics obtained by this approach and the Euler-Poincaré reduced system, we perform the Legendre transform and apply a well known theorem [26] to produce two equivalent Poisson brackets for the same set of equations. Moreover, we present the explicit form of the helicity invariant for nematic liquid crystals, as well as a whole class of Casimirs functionals for two dimensional incompressible flows.

2 Euler-Poincaré equations and Lie-Poisson brackets

2.1 The Euler-Poincaré approach

Let G be a Lie group acting on the *left* on a manifold M. We denote by $n \mapsto gn$ the action of $g \in G$ on $n \in M$. Then G acts naturally on the *right* on $TG \times M$ via the free action

$$(v_g, n) \mapsto (v_g h, h^{-1} n)\,,$$

where the action on the first factor is given by tangent lift of right translation on G, that is we denote $v_g h := TR_h(v_g)$, where TR_h is the tangent map to the right translation R_h by the element $h \in G$. Similarly, we will denote by $\alpha_g h$ the cotangent lifted action of right translation by an element $h \in G$, that is, we have $\alpha_g h := T^*R_{h^{-1}}(\alpha_g)$ where evidently $\alpha_g \in T^*_g G$. The quotient space $(TG \times M)/G$ can be identified with $\mathfrak{g} \times M$ via the diffeomorphism

$$[v_g, n] \mapsto (v_g g^{-1}, gn). \tag{2.1}$$

- Assume that we have a function $L : TG \times M \to \mathbb{R}$ which is *right* G-invariant.
- In particular, if $n_0 \in M$, define the Lagrangian

$$L_{n_0} : TG \to \mathbb{R}, \quad L_{n_0}(v_g) := L(v_g, n_0). \tag{2.2}$$

  Then L_{n_0} is *right* invariant under the lift to TG of the right action of G_{n_0} on G, where G_{n_0} is the isotropy group of n_0.
- Right G-invariance of L allows us to define $l : \mathfrak{g} \times M \to \mathbb{R}$ by

$$l(v_g g^{-1}, gn) = L(v_g, n). \tag{2.3}$$

- For a curve $g_t \in G$, let $\xi_t := \dot{g}_t g_t^{-1}$ and define the curve $n_t \in M$ as the unique solution of the following differential equation with time dependent coefficients

$$\dot{n}_t = (\xi_t)_M(n_t),$$

with initial condition n_0. Here $(\xi_t)_M \in \mathfrak{X}(M)$ denotes the infinitesimal generator associated to the time dependent Lie algebra element $\xi_t \in \mathfrak{g}$. The solution of this differential equation can be written as $n_t = g_t n_0$.

Theorem 2.1 *With the preceding notations, the following are equivalent:*

(i) *With $n_0 \in M$ held fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{n_0}(g, \dot{g}) dt = 0, \tag{2.4}$$

holds, for variations δg of g vanishing at the endpoints.

(ii) *g satisfies the Euler-Lagrange equations for L_{n_0} on G.*

(iii) *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi, n) dt = 0, \tag{2.5}$$

holds on $\mathfrak{g} \times M$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta n = \eta_M(n),$$

where $\eta \in \mathfrak{g}$ vanishes at the endpoints.

(iv) *The Euler-Poincaré equations hold on $\mathfrak{g} \times M$:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} = \mathbf{J}\left(\frac{\delta l}{\delta n}\right), \tag{2.6}$$

where $\mathbf{J} : T^*M \to \mathfrak{g}^*$, given by $\langle \mathbf{J}(\alpha_m), \xi \rangle = \langle \alpha_m, \xi_M(m) \rangle$, is the momentum map associated to the cotangent lifted action of G on T^*M.

The proof is an immediate generalization to that given in [22], in the case where M is a vector space on which G acts by a representation.

Remark 2.2 The Euler-Poincaré equations (2.6) can be rewritten in the form

$$\frac{\partial}{\partial t}\left[\mathrm{Ad}^*_g \frac{\delta l}{\delta \xi}\right] = \mathbf{J}\left(g^{-1}\frac{\delta l}{\delta n}\right). \tag{2.7}$$

Note that we have $g^{-1}\frac{\delta l}{\delta n} \in T^*_{n_0}M$, since $n_0 = g^{-1}n$.

Remark 2.3 (Relation to previous Euler-Poincaré formulations) In the special case when M is a vector space acted on by ordinary representations, we recover the formulation in [22], useful for heavy tops and compressible fluids. Also, when the vector space M is acted on by affine representations, the above picture recovers the results in [13], useful for complex fluids.

Remark 2.4 (Terminology) Throughout the paper, we will refer to equations (2.6) as *Euler-Poincaré equations* or *Euler-Poincaré equations for symmetry breaking* since they generalize the (*pure*) *Euler-Poincaré equations*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} = 0$$

on a Lie algebra (see [30]).

Remark 2.5 Since the Lagrangian $L_{n_0} : TG \to \mathbb{R}$ is G_{n_0}-invariant, it induces a reduced Lagrangian $\ell_{n_0} : TG/G_{n_0} \to \mathbb{R}$. Denoting by $i_{n_0} : TG \to TG \times M$ the injection $v_g \mapsto (v_g, n_0)$, we can write $L_{n_0} = L \circ i_{n_0}$. By passing to the quotient spaces, the injection i_{n_0} induces an injection

$$\bar{i}_{n_0} : TG/G_{n_0} \to \mathfrak{g} \times M, \quad [v_g]_{G_{n_0}} \mapsto (v_g g^{-1}, g n_0)$$

whose image is $\mathfrak{g} \times \mathrm{Orb}(n_0)$, where $\mathrm{Orb}(n_0) \subset M$ denotes the orbit of n_0. We have the relation $\ell_{n_0} = l \circ \bar{i}_{n_0}$ between the two reduced Lagrangians ℓ_{n_0} and l.

Remark 2.6 (Including translational motion in physical space) As we remarked in the introduction, the present treatment neglects translational motion in physical space Q. It is important to emphasize that this simplification is completely irrelevant. Indeed, translational motion may always be included a posteriori, upon extending the Lagrangian $L : TG \times M \to \mathbb{R}$ to be defined on the Cartesian product $TQ \times (TG \times M)$. Thus, by assuming that G acts trivially on Q, the new Lagrangian $L_{n_0}(q, \dot{q}, g, \dot{g})$ will produce the extra Euler-Lagrange equation on TQ, which is then accompanied by the Euler-Poincaré equation on $\mathfrak{g} \times M$. In conclusion, the reduced Lagrangian $l : TQ \times (\mathfrak{g} \times M)$ produces the equations

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \dot{q}} - \frac{\delta l}{\delta q} = 0\,, \qquad \frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} = \mathbf{J}\left(\frac{\delta l}{\delta n}\right),$$

where one has to remember the relation $\dot{n} = \xi_M(n)$.

2.2 The Lie-Poisson equations

Consider a function $H : T^*G \times M \to \mathbb{R}$ *right* invariant under the G-action on $T^*G \times M$ given by $(\alpha_g, n) \mapsto (\alpha_g h, h^{-1} n)$. In particular, the function $H_{n_0} : T^*G \to \mathbb{R}$ defined by $H_{n_0}(\alpha_g) = H(\alpha_g, n_0)$ is invariant under the induced action of the isotropy subgroup $G_{n_0} := \{g \in G \mid g n_0 = n_0\}$ for any $n_0 \in M$. The reduced Hamiltonian $h : \mathfrak{g}^* \times M \to \mathbb{R}$ is defined by $h(\alpha_g g^{-1}, gn) := H(\alpha_g, n)$.

Theorem 2.7 *Fix an element $n_0 \in M$. For $\alpha \in T^*_g G$ and $\mu := \alpha g^{-1} \in \mathfrak{g}^*$, the following are equivalent:*

i *The curve α satisfies Hamilton's equations for H_{n_0} on T^*G.*

ii *The curve* $(\mu, n) \in \mathfrak{g}^* \times M$ *is a solution of the Lie-Poisson equations*

$$\begin{cases} \dot{\mu} = -\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu - \mathbf{J}\left(\dfrac{\delta h}{\delta n}\right) \\ \dot{n} = \left(\dfrac{\delta h}{\delta \mu}\right)_M (n), \quad n(0) = n_0. \end{cases} \tag{2.8}$$

These equations are Hamiltonian relative to the Poisson bracket

$$\{f, g\}(\mu, n) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right] \right\rangle + \left\langle \mathbf{J}\left(\frac{\delta f}{\delta n}\right), \frac{\delta g}{\delta \mu} \right\rangle - \left\langle \mathbf{J}\left(\frac{\delta g}{\delta n}\right), \frac{\delta f}{\delta \mu} \right\rangle \tag{2.9}$$

on $\mathfrak{g}^* \times M$.

As on the Lagrangian side, the evolution of the variable n *is given by* $n = gn_0$.

Proof. Canonical Hamilton's equations for H_{n_0} on T^*G are equivalent to Hamilton's equation for H on $T^*G \times M$, endowed with the direct sum of the canonical Poisson bracket on T^*G and the zero Poisson bracket on M, with initial value n_0. Since H is G invariant, one can apply Poisson reduction to obtain the reduced Hamiltonian equations on the quotient manifold $(T^*G \times M)/G$ under the right action $(\alpha_g, n) \mapsto (\alpha_g h, h^{-1}n)$. We identify the quotient manifold $(T^*G \times M)/G$ with $\mathfrak{g}^* \times M$, via the diffeomorphism $[\alpha_g, n] \mapsto (\alpha_g g^{-1}, gn)$. Using Proposition 2.1 in [26], in the particular case where the Poisson structure is trivial on M, we obtain the reduced bracket (2.9). One then observes that the Hamilton's equations associated to this bracket are given by (2.8).

Legendre transformation. The preceding theorem is compatible with Theorem 2.1. Indeed, we can start with a Lagrangian $L_{n_0} : TG \to \mathbb{R}$ as in §2.1, that is, we have a function $L : TG \times M \to \mathbb{R}$ which is right G-invariant under the action $(v_g, n) \mapsto (v_g h, h^{-1}n)$, such that $L_{n_0}(v_g) = L(v_g, n_0)$. Then L_{n_0} is right invariant under the lift to TG of the right action of the isotropy group G_{n_0} on G. Suppose that the Legendre transformation $\mathbb{F}L_{n_0}$ is invertible and form the corresponding Hamiltonian $H_{n_0} = E_{n_0} \circ \mathbb{F}L_{n_0}^{-1}$, where E_{n_0} is the energy of L_{n_0}, see [30]. Then the function $H : T^*G \times M \to \mathbb{R}$ so defined is G-invariant and one can apply this theorem. At the level of the reduced space, to a reduced Lagrangian $l : \mathfrak{g} \times M \to \mathbb{R}$ we associate the reduced Hamiltonian $h : \mathfrak{g}^* \times M \to \mathbb{R}$ given by

$$h(\mu, n) := \langle \mu, \xi \rangle - l(\xi, n), \quad \mu = \frac{\delta l}{\delta \xi}.$$

Since

$$\frac{\delta h}{\delta \mu} = \xi \quad \text{and} \quad \frac{\delta h}{\delta n} = -\frac{\delta l}{\delta n},$$

we see that the Hamilton's equations for h on $\mathfrak{g} \times M$ are equivalent to the Euler-Poincaré equation (2.6) for l together with the kinematic equation

$$\dot{n} = \xi_M(n).$$

Again, when M is a vector space acted on by a linear or affine representation, the above Lie-Poisson reduction recovers the results in [32] and [13], respectively.

3 Transitive actions: the dynamics of nematic molecules

This section considers the interesting case of a Lie group G acting transitively from the *left* on a manifold M, called *order parameter space*. In this particular case we use the notation $G = \mathcal{O}$, since we think of G as the *order parameter group* or the *broken symmetry*.

This is a typical situation in condensed matter physics, which describes exactly the situation presented in the Introduction. The particular choice of the group $\mathcal{O}$ and the manifold M depends on the system under consideration. For example, the common case $\mathcal{O} = SO(3)$ and $M = \mathbb{R}P^2 \simeq S^2/\mathbb{Z}_2$ yields ordinary nematic particles, while $\mathcal{O} = SO(3)$ and $M = SO(3)/D_2$ yields biaxial nematics. Here D_2 is the dihedral group generated by 180°-rotations and reflections. Many other choices are certainly possible and we address the reader to [33, 34] for reviews on these topics.

Before proceeding, we emphasize that the present approach focuses on the rotational dynamics of a fixed nematic particle (e.g. ordinary or biaxial) in an external potential. Although this situation is physically improbable, its extension to more realistic configurations is straightforward. For example, translational motion in physical space Q can be included as discussed previously, while the dynamics of N interacting particles requires extending the present treatment to the space $Q^N \times (\mathcal{O} \times \mathcal{O}/\mathcal{P})^N$. The continuum limit of a nematic lattice and the full hydrodynamic model are analyzed in section 5.

As we have shown in the Introduction, if we fix a reference point $n_0 \in M$ and consider the isotropy group $\mathcal{P} = \mathcal{O}_{n_0} = \{\psi \in \mathcal{O} \mid \psi n_0 = n_0\}$, then the orbit map

$$\mathcal{O} \to M, \quad \chi \mapsto \chi n_0$$

induces a diffeomorphism

$$\Psi^{n_0} : \mathcal{O}/\mathcal{P} \to M, \quad [\chi] = \chi\mathcal{P} \mapsto \chi n_0 \,.$$

The reference point n_0 corresponds to the coset $[e] = e\mathcal{P} = \mathcal{P}$ of $\mathcal{O}/\mathcal{P}$. Since the action is transitive, given a reference point n_0, a $\mathcal{P}$-invariant Lagrangian $L_{n_0} : T\mathcal{O} \to \mathbb{R}$ completely determines the right invariant function $L : T\mathcal{O} \times (\mathcal{O}/\mathcal{P}) \to \mathbb{R}$. Indeed, it suffices to define $L(v_\chi, n) := L_{n_0}(v_\chi\psi)$, where $\psi \in \mathcal{O}$ is such that $n = \psi n_0$, by transitivity of the action. In particular, L_{n_0} determines the Lagrangians L_{m_0} for all other reference points $m_0 \in M$. Note also that the injection $\bar{i}_{n_0}$ is here a diffeormorphism since $\mathrm{Orb}(n_0) = M$, that is, we have the diffeomorphism

$$\bar{i}_{n_0} : T\mathcal{O}/\mathcal{P} \to \mathfrak{o} \times M, \quad [v_\chi] \mapsto (v_\chi\chi^{-1}, \chi n_0). \tag{3.1}$$

3.1 Dynamics of uniaxial nematic molecules

Theorem 2.1 applies to nematic particles without changes. Thus, in this case $\mathcal{O} = SO(3)$, $\mathcal{O}/\mathcal{P} = \mathbb{R}P^2$ and we have the following Euler-Poincaré equations arising from the reduced Lagrangian $l : \mathfrak{so}(3) \times \mathbb{R}P^2 \to \mathbb{R}$:

$$\frac{d}{dt}\frac{\delta l}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \frac{\delta l}{\delta \boldsymbol{\nu}} + \mathbf{n} \times \frac{\delta l}{\delta \mathbf{n}}\,, \qquad \dot{\mathbf{n}} = \boldsymbol{\nu} \times \mathbf{n} \tag{3.2}$$

where we have confused the space $\mathbb{R}P^2$ of unsigned unit vectors with ordinary unit vectors in $\mathbb{R}^3$, thereby emphasizing the strict relation between rotational motion of nematic particles and the heavy top dynamics. Indeed, it is easier to consider the Lagrangian $l : \mathfrak{so}(3) \times (S^2/\mathbb{Z}_2) \to \mathbb{R}$ as

a function $l : \mathfrak{so}(3) \times S^2 \to \mathbb{R}$ possessing the $\mathbb{Z}_2$-symmetry $l(\boldsymbol{\nu}, \mathbf{v}) = l(\boldsymbol{\nu}, -\mathbf{v})$, for all $\boldsymbol{\nu} \in \mathfrak{so}(3)$ and $\mathbf{v} \in S^2$. This is how equations (3.2) are written: one chooses a representative vector for $\mathbf{n}$ to compute the equations (i.e. a unit vector $\mathbf{v} \in S^2$ such that $[\mathbf{v}] = \mathbf{n}$), and then checks that the result does not depend on the chosen representative.

The Lagrangian for the nematic particle is $L_{\mathbf{n}_0} : TSO(3) \to \mathbb{R}$

$$L_{\mathbf{n}_0}(\chi, \dot{\chi}) = \frac{1}{2} j \, |\dot{\chi}|^2 - \Phi_{\mathbf{n}_0}(\chi),$$

where $j > 0$ is the moment of inertia and $|\cdot|$ denotes the norm associated to the $SO(3)$-invariant Riemannian metric $\langle \dot{\chi}_1, \dot{\chi}_2 \rangle = \mathrm{Tr}(\dot{\chi}_1^T \dot{\chi}_2)$ on $SO(3)$. We notice that the inertia tensor $\mathbb{I} = j\,\mathbf{I}$ leads to the schematic description of a nematic molecule in terms of a spherical rigid body endowed with a special direction $\mathbf{n}_0$. The unreduced potential energy is $\Phi_{\mathbf{n}_0} : SO(3) \to \mathbb{R}$. An example of such an external potential for a nematic particle is given by the quadratic expression: $\Phi_{\mathbf{n}_0}(\chi) = \lambda \, |\mathbf{n}_0 \cdot \chi^{-1}\mathbf{k}|^2/2$, where $\lambda \in \mathbb{R}$ is a constant parameter and $\mathbf{k}$ plays the role of an external force field, e.g. an external magnetic (or electric) field. (Another possibility could be $\Phi_{\mathbf{n}_0}(\chi) = \lambda \, |\mathbf{n}_0 \times (\chi^{-1}\mathbf{k})|^2/2)$. Besides the quadratic form of the potential, the important difference from the heavy-top case is that we now identify $\mathbf{n}_0$ with $-\mathbf{n}_0$ and the same for the fixed vector $\mathbf{k}$, which thus can point upwards or downwards without distinction. The presence of the potential $\Phi_{\mathbf{n}_0}$ breaks the symmetry, since $L_{\mathbf{n}_0}$ is only $O(2)$-invariant (or better D_∞-invariant, as mentioned in the Introduction) under right translation. If $\mathbf{n}_0 \in S^2/\mathbb{Z}_2$ is considered as an arbitrary director and we define $L : TSO(3) \times (S^2/\mathbb{Z}_2) \to \mathbb{R}$ by $L(\chi, \dot{\chi}, \mathbf{n}_0) := L_{\mathbf{n}_0}(\chi, \dot{\chi})$, then L is $SO(3)$-invariant under the right action $(\chi, \dot{\chi}, \mathbf{n}_0) \mapsto (\chi\psi, \dot{\chi}\psi, \psi^{-1}\mathbf{n}_0)$. The argument is the same in the heavy-top case, as shown by the following calculation:

$$\Phi(\chi, \mathbf{n}_0) = \frac{\lambda}{2} \left|\mathbf{n}_0 \cdot \chi^{-1}\mathbf{k}\right|^2 = \frac{\lambda}{2} \left|\chi\mathbf{n}_0 \cdot \mathbf{k}\right|^2 = \Phi(\chi\chi^{-1}, \chi\mathbf{n}_0) =: \phi(\chi\mathbf{n}_0)$$

where we have introduced the reduced potential $\phi : S^2/\mathbb{Z}_2 \to \mathbb{R}$. Thus, by $SO(3)$-invariance, L induces the reduced Lagrangian l governing the rotational dynamics of a single nematic particle:

$$l(\boldsymbol{\nu}, \mathbf{n}) = \frac{1}{2} \, j \, |\boldsymbol{\nu}|^2 - \phi(\mathbf{n}) = \frac{1}{2} \, j \, |\boldsymbol{\nu}|^2 - \frac{\lambda}{2} \, |\mathbf{n} \cdot \mathbf{k}|^2 \tag{3.3}$$

where $\boldsymbol{\nu} \in \mathfrak{so}(3)$, and $\mathbf{n} \in S^2/\mathbb{Z}_2$. In this case, the Euler-Poincaré equations for nematics reads

$$j\dot{\boldsymbol{\nu}} = \nabla\phi(\mathbf{n}) \times \mathbf{n} = \lambda \, |\mathbf{n} \cdot \mathbf{k}| \, \mathbf{k} \times \mathbf{n}, \quad \dot{\mathbf{n}} = \boldsymbol{\nu} \times \mathbf{n} \tag{3.4}$$

by (3.2). Moreover, the Legendre transform $\boldsymbol{\mu} = \delta l / \delta \boldsymbol{\nu}$ produces the reduced quadratic Hamiltonian on $\mathfrak{so}(3)^* \times \mathbb{R}P^2$:

$$h(\boldsymbol{\mu}, \mathbf{n}) = \frac{1}{2j} |\boldsymbol{\mu}|^2 + \frac{\lambda}{2} \, |\mathbf{n} \cdot \mathbf{k}|^2 \tag{3.5}$$

The corresponding Lie-Poisson equations are

$$\frac{d}{dt}\boldsymbol{\mu} = \frac{\delta h}{\delta \boldsymbol{\mu}} \times \boldsymbol{\mu} + \frac{\delta h}{\delta \mathbf{n}} \times \mathbf{n} \, , \qquad \dot{\mathbf{n}} = \frac{\delta h}{\delta \boldsymbol{\mu}} \times \mathbf{n}$$

while the Lie-Poisson bracket of functions $f, g \in \mathcal{F}(\mathfrak{so}(3)^* \times \mathbb{R}P^2)$ is

$$\{f, g\}(\boldsymbol{\mu}, \mathbf{n}) = \boldsymbol{\mu} \cdot \left(\frac{\delta f}{\delta \boldsymbol{\mu}} \times \frac{\delta g}{\delta \boldsymbol{\mu}} \right) + \mathbf{n} \cdot \left(\frac{\delta f}{\delta \mathbf{n}} \times \frac{\delta g}{\delta \boldsymbol{\mu}} - \frac{\delta g}{\delta \mathbf{n}} \times \frac{\delta f}{\delta \boldsymbol{\mu}} \right). \tag{3.6}$$

A special case of physical interest in microscopic theories of liquid crystals is the case of an ensemble of N interacting nematic particles moving in the physical space Q. In this case, the unreduced Lagrangian $L_{\mathbf{n}_1^{(0)},\dots,\mathbf{n}_N^{(0)}} : TQ^N \times TSO(3)^N \to \mathbb{R}$ is given by

$$L_{\mathbf{n}_1^{(0)},\dots,\mathbf{n}_N^{(0)}}(\chi_1,\dots,\chi_N) = \frac{1}{2}\sum_{i=1}^{N}\|\dot{q}_i\|^2 + \frac{1}{2}\,j\sum_{i=1}^{N}|\dot{\chi}_i|^2 - \sum_{i\neq k}\Phi_{\mathbf{n}_1^{(0)},\dots,\mathbf{n}_N^{(0)}}(q_i,q_k,\chi_i,\chi_k),$$

where the first norm is associated to a Riemannian metric on Q, while $\mathcal{O} = SO(3)^N$ is endowed with the direct product group structure. The order parameter space is now $M = (S^2/\mathbb{Z}_2)^N$. As before, Euler-Poincaré reduction yields the reduced Lagrangian $l : TQ^N \times (\mathfrak{so}(3) \times S^2/\mathbb{Z}_2)^N \to \mathbb{R}$, which is written as

$$l(q_i,\boldsymbol{\nu}_i,\mathbf{n}_i) = \frac{1}{2}\sum_{i=1}^{N}\|\dot{q}_i\|^2 + \frac{1}{2}\,j\sum_{i=1}^{N}|\boldsymbol{\nu}_i|^2 - \sum_{i\neq k}\phi(q_i,q_k,\mathbf{n}_i,\mathbf{n}_k) \tag{3.7}$$

and produces Euler-Lagrange equations on TQ^N and Euler-Poincaré equations on $(\mathfrak{so}(3)\times\mathbb{R}P^2)^N$. This approach is adopted in the physics literature to formulate microscopic approaches for nematic liquid crystals [35, 28, 2]. In this framework, when $Q = \mathbb{R}^n$, the reduced potential is usually expressed as (cf. [28])

$$\phi(q_i,q_k,\mathbf{n}_i,\mathbf{n}_k) = V(q_i-q_k) + W(q_i-q_k)\,|\mathbf{n}_i\cdot\mathbf{n}_k|^2\,.$$

where the angular factor $|\mathbf{n}_i\cdot\mathbf{n}_k|^2$ is reminiscent of the well known Maier-Saupe potential, in alternative to Onsager's expression $|\mathbf{n}_i\times\mathbf{n}_k|^2$. In what follows, however, we shall pursue the question of how rotational motion is affected by symmetry breaking, without considering translational motion in physical space, since the latter can be always considered a posteriori, once the reduction has been carried out on the rotational part of the Lagrangian.

Remark 3.1 (The heavy top revisited) The same Euler-Poincaré approach also applies to the simpler case when the order parameter manifold is $M = S^2$. In the case of symmetry breaking, the Lagrangian $L : TSO(3) \to \mathbb{R}$ is not $SO(3)$-invariant. This is exactly the heavy-top case (see equation (1.1)), which involves a Lagrangian L that is only $SO(2)$-invariant (rotations with respect to the vertical axis). The transitivity of the $SO(2)$-action on $SO(3)$ recovers the well known isomorphism $S^2 = SO(3)/SO(2)$, thereby producing a reduced Lagrangian $l : \mathfrak{so}(3)\times(SO(3)/SO(2)) \to \mathbb{R}$. This approach recovers the well known treatment of heavy top dynamics (cf. e.g. [22]), avoiding the necessity of identify points in S^2 with vectors in $\mathbb{R}^3$.

3.2 Alignment tensor dynamics for uniaxial nematics

Although the order parameter of a nematic liquid crystal often considered as a director, such a quantity is also difficult to work with analytically. In fact, the Landau-de Gennes theory typically involves the components of a symmetric matrix Q, usually known as alignment tensor. In matrix notation, this tensor is given by (cf. e.g. [6])

$$\mathsf{Q} = \frac{1}{2}\left(\mathbf{n}\mathbf{n}^T - \frac{1}{3}\,\mathbf{I}\right) \tag{3.8}$$

where $\mathbf{n}\mathbf{n}^T$ is also denoted as the dyadic form $\mathbf{n}\mathbf{n}$. In order to write the Poisson bracket (3.6) in terms of the Q, we first need to compute functional derivatives. This is done by simply imposing the equality $\delta f(\boldsymbol{\sigma}, \mathbf{n}) = \delta f(\boldsymbol{\sigma}, \mathsf{Q})$, which yields the relation

$$\frac{\delta f}{\delta \mathbf{n}} = \mathbf{n}^T \frac{\delta f}{\delta \mathsf{Q}}$$

At this point, it suffices to express the following term in terms of the alignment tensor:

$$\begin{aligned}\mathbf{n} \cdot \frac{\delta f}{\delta \boldsymbol{\mu}} \times \frac{\delta g}{\delta \mathbf{n}} &= -\operatorname{Tr}\left(\mathbf{n}^T \frac{\delta g}{\delta \mathsf{Q}} \frac{\delta f}{\delta \hat{\boldsymbol{\mu}}} \mathbf{n}\right) = -\operatorname{Tr}\left(\left(2\mathsf{Q} + \frac{1}{3}\mathbf{I}\right) \frac{\delta g}{\delta \mathsf{Q}} \frac{\delta f}{\delta \hat{\boldsymbol{\mu}}}\right) \\ &= -2\operatorname{Tr}\left(\mathsf{Q} \frac{\delta g}{\delta \mathsf{Q}} \frac{\delta f}{\delta \hat{\boldsymbol{\mu}}}\right) = -\operatorname{Tr}\left(\left[\mathsf{Q}, \frac{\delta g}{\delta \mathsf{Q}}\right] \frac{\delta f}{\delta \hat{\boldsymbol{\mu}}}\right) = \operatorname{Tr}\left(\mathsf{Q}\left[\frac{\delta f}{\delta \hat{\boldsymbol{\mu}}}, \frac{\delta g}{\delta \mathsf{Q}}\right]\right)\end{aligned}$$

where we have introduced the antisymmetric hat matrix $\hat{\mu}_{ij} = \varepsilon_{ijk}\mu_k$ and we have used the fact that Q is symmetric. Thus, the bracket (3.6) becomes

$$\{f, g\}(\boldsymbol{\mu}, \mathbf{n}) = \boldsymbol{\mu} \cdot \left(\frac{\delta f}{\delta \boldsymbol{\mu}} \times \frac{\delta g}{\delta \boldsymbol{\mu}}\right) + \operatorname{Tr}\left(\mathsf{Q}\left(\left[\frac{\delta f}{\delta \hat{\boldsymbol{\mu}}}, \frac{\delta g}{\delta \mathsf{Q}}\right] - \left[\frac{\delta g}{\delta \hat{\boldsymbol{\mu}}}, \frac{\delta f}{\delta \mathsf{Q}}\right]\right)\right), \tag{3.9}$$

which is the Lie-Poisson structure associated to the semidirect product $SO(3) \circledS \operatorname{Sym}(3)$, relative to the representation $\mathsf{Q} \mapsto \chi \mathsf{Q} \chi^{-1}$. Consequently, we have shown that the map $(\boldsymbol{\mu}, \mathbf{n}) \mapsto (\boldsymbol{\mu}, \mathsf{Q}) \in \mathfrak{so}^*(3) \times \operatorname{Sym}(3)$ is Poisson relative to the Lie-Poisson brackets (3.6) and (3.9). This result should not come as surprise, since the map $\mathbf{n} \mapsto \mathsf{Q}$ is equivariant with respect to the $SO(3)$ actions $\mathbf{n} \mapsto \chi \mathbf{n}$ and $\mathsf{Q} \mapsto \chi \mathsf{Q} \chi^{-1}$.

The resulting Lie-Poisson equations on $\mathfrak{so}^*(3) \times \operatorname{Sym}(3)$ are

$$\frac{d}{dt}\boldsymbol{\mu} = \frac{\delta h}{\delta \boldsymbol{\mu}} \times \boldsymbol{\mu} + \overrightarrow{\left[\mathsf{Q}, \frac{\delta h}{\delta \mathsf{Q}}\right]}, \qquad \dot{\mathsf{Q}} = \left[\frac{\delta h}{\delta \hat{\boldsymbol{\mu}}}, \mathsf{Q}\right]$$

with the notation $\overrightarrow{A}_i = \varepsilon_{ijk} A_{jk}$ and the following Hamiltonian (up to an irrelevant constant) arising from its previous expressions (3.5):

$$h(\boldsymbol{\mu}, Q) = \frac{1}{2j}|\boldsymbol{\mu}|^2 + \frac{\lambda}{2}\mathbf{k}^T Q\, \mathbf{k}\,. \tag{3.10}$$

Notice that more general potential terms of the type $\mathbf{k}_1^T Q\, \mathbf{k}_2$ are also allowed, which correspond to $(\mathbf{n} \cdot \mathbf{k}_1)(\mathbf{n} \cdot \mathbf{k}_2)$ in (3.5). However, in the Landau-de Gennes theory, the potential energy $\phi(\mathsf{Q})$ involves more complicated expansions of the form

$$\phi(\mathsf{Q}) = \sum_{s,n \geq 1} a_{sn} \left(\operatorname{Tr}(\mathsf{Q}^n)\right)^s$$

where the coefficients a_{sn} are physical constants, see [8] .

At this point, it is important to notice that the Euler-Poincaré and Lie-Poisson equations involving the alignment tensor Q can also be derived directly by reduction of the Lagrangian $L_{\mathsf{Q}_0}(\chi, \dot{\chi})$ or the Hamiltonian $H_{\mathsf{Q}_0}(\chi, \Psi)$ (with $(\chi, \Psi) \in T^*SO(3)$), for a given $\mathsf{Q}_0 \in \operatorname{Sym}(3)$. This process is independent of the definition of the director variable $\mathbf{n}$.

3.3 Dynamics of biaxial nematic molecules

The same treatment applies to the case of biaxial nematic particles. In this context, we can identify the order parameter space with the manifold M of ordered couples $n = (\mathbf{n}_1, \mathbf{n}_2)$ of mutually orthogonal, unsigned unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$ (cf. e.g. [29]). This choice is consistent with the microstructure of a biaxial molecule, which can thus be envisioned as a particle carrying two orthogonal director variables determining a special rotational state. The group $SO(3)$ acts transitively on the ordered couples in M by matrix multiplication on each director: $(\mathbf{n}_1, \mathbf{n}_2) \mapsto (\chi\mathbf{n}_1, \chi\mathbf{n}_2)$. In order to determine the nature of symmetry breaking for biaxial particles, we can fix a reference couple $n_0 \in M$: for example, one considers $((1,0,0),(0,1,0))$. Then, the associated isotropy subgroup $SO(3)_{n_0} \subset SO(3)$ is readily seen to be the dihedral group

$$D_2 = \{\mathrm{diag}(1,1,1), \mathrm{diag}(-1,-1,1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(1,-1,-1)\} = SO(3)_{n_0}\,.$$

Therefore we can write $M = SO(3)/D_2$ and we can express the Euler-Poincaré equations for a biaxial nematic particle as

$$\frac{d}{dt}\frac{\delta l}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \frac{\delta l}{\delta \boldsymbol{\nu}} + \mathbf{n}_i \times \frac{\delta l}{\delta \mathbf{n}_i}\,, \qquad \dot{\mathbf{n}}_i = \mathbf{n}_i \times \boldsymbol{\nu} \tag{3.11}$$

where we assume summation over repeated indexes in the first equation. A straightforward calculation may verify that the orthogonality condition $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ is consistently preserved at all times. A simple form of Lagrangian for a single biaxial nematic particle is given by

$$l(\boldsymbol{\nu}, \mathbf{n}_1, \mathbf{n}_2) = \frac{1}{2}\, j\, |\boldsymbol{\nu}|^2 - \frac{1}{2}\Big(\lambda_1 \left|\mathbf{n}_1 \cdot \mathbf{k}_1\right|^2 + \lambda_2 \left|\mathbf{n}_2 \cdot \mathbf{k}_2\right|^2 + \lambda_3 \left|\mathbf{n}_1 \times \mathbf{n}_2 \cdot \mathbf{k}_2\right|^2\Big)$$

where $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ again play the role of an external force field. A simple Legendre transform of equations (3.11) yields the following Poisson bracket for biaxial particles:

$$\begin{aligned}\{f,g\}(\boldsymbol{\mu}, \mathbf{n}_1, \mathbf{n}_2) = \boldsymbol{\mu} \cdot \left(\frac{\delta f}{\delta \boldsymbol{\mu}} \times \frac{\delta g}{\delta \boldsymbol{\mu}}\right) + \mathbf{n}_1 \cdot \left(\frac{\delta f}{\delta \mathbf{n}_1} \times \frac{\delta g}{\delta \boldsymbol{\mu}} - \frac{\delta g}{\delta \mathbf{n}_1} \times \frac{\delta f}{\delta \boldsymbol{\mu}}\right) \\ + \mathbf{n}_2 \cdot \left(\frac{\delta f}{\delta \mathbf{n}_2} \times \frac{\delta g}{\delta \boldsymbol{\mu}} - \frac{\delta g}{\delta \mathbf{n}_2} \times \frac{\delta f}{\delta \boldsymbol{\mu}}\right).\end{aligned} \tag{3.12}$$

The next section shows how the above bracket can be written in terms of the so called alignment tensors, analogously to our previous discussion concerning ordinary nematic molecules.

Remark 3.2 (V-shaped molecules) The orthogonality condition $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ can be weakened by considering more general configurations of V-shaped molecules carrying two directors spanning a fixed angle ϑ. In this case, the isotropy subgroup is not the dihedral group D_2. Rather, it is formed of π-rotations only, and thus it can be identified with the cyclic group $\mathbb{Z}_2$. In this case, the order parameter manifold is $M = SO(3)/\mathbb{Z}_2$ and the equations (3.11) are still valid, thereby describing the dynamics of a single V-shaped molecule with directors $\mathbf{n}_1$ and $\mathbf{n}_2$, such that $\mathbf{n}_1 \cdot \mathbf{n}_2 = \cos\vartheta$ at all times.

3.4 Alignment tensor dynamics for biaxial nematics

In analogy with ordinary nematic molecules, we now show how one can write the Lie-Poisson equations for biaxial molecules in terms of the alignment tensor which is usually given by a linear combination of the two symmetric matrices

$$\mathsf{A} = \frac{1}{2}\left(\mathbf{n}_1\mathbf{n}_1^T - \frac{1}{3}\mathbf{I}\right), \qquad \mathsf{B} = \frac{1}{2}\left(\mathbf{n}_2\mathbf{n}_2^T - (\mathbf{n}_1\times\mathbf{n}_2)(\mathbf{n}_1\times\mathbf{n}_2)^T\right). \tag{3.13}$$

Following [37], we shall begin by considering the above matrices as two separate variables, so that the equality $\delta f(\boldsymbol{\sigma},\mathbf{n}_1,\mathbf{n}_2) = \delta f(\boldsymbol{\sigma},\mathsf{A},\mathsf{B})$ yields the relations

$$\begin{aligned}
\frac{\delta f}{\delta \mathbf{n}_1} &= \mathbf{n}_1^T\frac{\delta f}{\delta \mathsf{A}} - \mathbf{n}_2\times(\mathbf{n}_1\times\mathbf{n}_2)^T\frac{\delta f}{\delta \mathsf{B}}\\
\frac{\delta f}{\delta \mathbf{n}_2} &= \mathbf{n}_2^T\frac{\delta f}{\delta \mathsf{B}} + \mathbf{n}_1\times(\mathbf{n}_1\times\mathbf{n}_2)^T\frac{\delta f}{\delta \mathsf{B}}
\end{aligned}$$

In analogy to the procedure that we have followed for the case of ordinary nematic particles, we now express the following quantity in terms of (A,B):

$$\begin{aligned}
\mathbf{n}_1\cdot\frac{\delta f}{\delta\boldsymbol{\mu}}\times\frac{\delta g}{\delta\mathbf{n}_1} + \mathbf{n}_2\cdot\frac{\delta f}{\delta\boldsymbol{\mu}}\times\frac{\delta g}{\delta\mathbf{n}_2} =& -\frac{\delta f}{\delta\boldsymbol{\mu}}\cdot\mathbf{n}_1\times\left(\mathbf{n}_1^T\frac{\delta f}{\delta\mathsf{A}} - \mathbf{n}_2\times(\mathbf{n}_1\times\mathbf{n}_2)^T\frac{\delta f}{\delta\mathsf{B}}\right)\\
&-\frac{\delta f}{\delta\boldsymbol{\mu}}\cdot\mathbf{n}_2\times\left(\mathbf{n}_2^T\frac{\delta f}{\delta\mathsf{B}} + \mathbf{n}_1\times(\mathbf{n}_1\times\mathbf{n}_2)^T\frac{\delta f}{\delta\mathsf{B}}\right)\\
=& -\frac{\delta f}{\delta\boldsymbol{\mu}}\cdot\left(\mathbf{n}_1\times\mathbf{n}_1^T\frac{\delta f}{\delta\mathsf{A}} + \mathbf{n}_2\times\mathbf{n}_2^T\frac{\delta f}{\delta\mathsf{B}}\right.\\
&\qquad\left. -(\mathbf{n}_1\times\mathbf{n}_2)\times(\mathbf{n}_1\times\mathbf{n}_2)^T\frac{\delta f}{\delta\mathsf{B}}\right)\\
=& \ \mathrm{Tr}\left(\mathsf{A}\left[\frac{\delta f}{\delta\hat{\mu}},\frac{\delta g}{\delta\mathsf{A}}\right]\right) - \mathrm{Tr}\left(\mathbf{n}_2\mathbf{n}_2^T\frac{\delta g}{\delta\mathsf{B}}\frac{\delta f}{\delta\hat{\boldsymbol{\mu}}}\right)\\
&+\mathrm{Tr}\left((\mathbf{n}_1\times\mathbf{n}_2)(\mathbf{n}_1\times\mathbf{n}_2)^T\frac{\delta g}{\delta\mathsf{B}}\frac{\delta f}{\delta\hat{\boldsymbol{\mu}}}\right)\\
=& \ \mathrm{Tr}\left(\mathsf{A}\left[\frac{\delta f}{\delta\hat{\boldsymbol{\mu}}},\frac{\delta g}{\delta\mathsf{A}}\right]\right) + \mathrm{Tr}\left(\mathsf{B}\left[\frac{\delta f}{\delta\hat{\boldsymbol{\mu}}},\frac{\delta g}{\delta\mathsf{B}}\right]\right),
\end{aligned}$$

where we have used the same formulas that emerged in the case of nematic particles. Therefore, we can write the Poisson bracket (3.12) for biaxial particles in terms of the tensor order parameters as

$$\begin{aligned}
\{f,g\}(\boldsymbol{\mu},\mathsf{A},\mathsf{B}) = \boldsymbol{\mu}\cdot\left(\frac{\delta f}{\delta\boldsymbol{\mu}}\times\frac{\delta g}{\delta\boldsymbol{\mu}}\right) &+ \mathrm{Tr}\left(\mathsf{A}\left(\left[\frac{\delta f}{\delta\hat{\boldsymbol{\mu}}},\frac{\delta g}{\delta\mathsf{A}}\right] - \left[\frac{\delta g}{\delta\hat{\boldsymbol{\mu}}},\frac{\delta f}{\delta\mathsf{A}}\right]\right)\right)\\
&+\mathrm{Tr}\left(\mathsf{B}\left(\left[\frac{\delta f}{\delta\hat{\boldsymbol{\mu}}},\frac{\delta g}{\delta\mathsf{B}}\right] - \left[\frac{\delta g}{\delta\hat{\boldsymbol{\mu}}},\frac{\delta f}{\delta\mathsf{B}}\right]\right)\right).
\end{aligned} \tag{3.14}$$

At this point, it is easy to see that any linear combination of the form $\mathsf{Q} = \alpha\mathsf{A}+\beta\mathsf{B}$ leads precisely to the bracket (3.9). Indeed, this follows upon noticing that

$$\frac{\delta f}{\delta\mathsf{A}} = \alpha\frac{\delta f}{\delta\mathsf{Q}}, \qquad \frac{\delta f}{\delta\mathsf{B}} = \beta\frac{\delta f}{\delta\mathsf{Q}}$$

and by replacing the above functional derivatives in the bracket (3.14).

Remark 3.3 (Other order parameter spaces) The present Euler-Poincaré approach applies to any sort of order parameter manifold. For example, in superfluid dynamics one is faced to more complicated coset structures such as $(SO(3) \times SO(3)) / (SO(2) \times \mathbb{Z}_2)$, which is the order parameter space for the B-phase of superfluid Helium-3 [34]. Another interesting example is provided by ^{3}He-A, i.e. the A-phase of ^{3}He, whose order parameter space is $(SO(3) \times SO(3)) / SO(3) = SO(3)$. In the context of liquid crystals one of the most complicated examples is provided by smectics, whose order parameter space involves the special Euclidean group [34]. Other possibilities still include more general complex fluids, such as micromorphic or micropolar fluids [10]. See [13] for a geometric treatment similar to the present one.

Remark 3.4 (A fully symmetric case: magnetic moment dynamics) All the above examples are indicative of how often symmetry breaking appears in physics. However, one should not forget that fully symmetric Lie-Poisson systems also emerge in condensed matter applications. A famous example is provided by the magnetic moments of ferromagnetic media, whose evolution takes place on coadjoint orbits of the rotation group $SO(3)$. In this sense, the dynamics of the electron magnetic moment $\dot{\boldsymbol{\mu}} = \gamma \boldsymbol{\mu} \times \mathbf{H}$ (where $\mathbf{H}$ is the external magnetic field and γ is a physical constant) is a Lie-Poisson system on $\mathfrak{so}^*(3)$ with Hamiltonian $h(\boldsymbol{\mu}) = \gamma \boldsymbol{\mu} \cdot \mathbf{H}$. Thus, the magnetic moment dynamics possesses the same geometric interpretation as the rigid body dynamics, with the only difference that the latter also determines a geodesic flow on $SO(3)$. A posteriori, one selects a special coadjoint orbit (the unit sphere S^2) to be consistent with the definition of spin as a unit vector. Indeed, magnetic moments possess precisely the same symmetry properties of the usual spin variable, whose dynamics governs the theory of spin glasses [21].

4 Lagrange-Poincaré approach to symmetry breaking

While the last two sections presented the Euler-Poincaré and Lie-Poisson formulations for systems with symmetry breaking, the present section will show how the same systems allow for another geometric description, whose underlying general theory is known under the name of Lagrangian reduction [3, 5]. While the Euler-Poincaré theory applies to systems possessing a Lie group as the configuration space, Lagrangian reduction may be used to approach any system on a tangent bundle possessing a continuous symmetry. As we shall see, the resulting Lagrange-Poincaré equations produce a special variable possessing a purely geometric character, and whose physical nature is analogue to the color charge in Yang-Mills theories. In these theories, the motion of a colored particle in a Yang-Mills field is a trajectory on a principal bundle B determined by an invariant Hamiltonian on T^*B, where the Poisson bracket is canonical. In the particular case when the Hamiltonian is quadratic, the resulting Wong's geodesic equations produce the well known coadjoint dynamics for the color charge. It is clear that for Abelian Yang-Mills theories, the color is constant, like in the case of electromagnetism (cf. [30]). We shall see how a very similar situation also appears for the dynamics of nematic particles, whose celebrated Ericksen-Leslie equations are equivalent to the Euler-Lagrange equations on $T\mathbb{R}P^2$.

4.1 General Lagrangian reduction: a brief review

Consider a *right* action $\Phi : G \times Q \to Q$ of a Lie group G on a manifold Q. Let $L : TQ \to \mathbb{R}$ be a G-invariant Lagrangian under the cotangent-lifted action of G on TQ. Because of this invariance, we get a well defined reduced Lagrangian $\ell : (TQ)/G \to \mathbb{R}$ satisfying $\ell([v_q]) = L(v_q)$. Assuming the group action is free and proper, the quotient space $(TQ)/G$ is intrinsically a vector bundle over $T(Q/G)$ with a fiber modeled on the Lie algebra $\mathfrak{g}$. Using a connection $\mathcal{A}$ on the principal bundle $\pi : Q \to Q/G$ we have a vector bundle isomorphism

$$\alpha_{\mathcal{A}} : (TQ)/G \longrightarrow T(Q/G) \oplus \tilde{\mathfrak{g}}, \quad [v_q] \longmapsto \alpha_{\mathcal{A}}([v_q]) := (T\pi(v_q), [q, \mathcal{A}(v_q)]_G)$$

over Q/G, where the associated bundle $\tilde{\mathfrak{g}} := Q \times_G \mathfrak{g}$, is defined as the quotient space of $Q \times \mathfrak{g}$ relative to the right action $(q, \xi) \mapsto (\Phi_g(q), \mathrm{Ad}_{g^{-1}} \xi)$ of G. The elements of $\tilde{\mathfrak{g}}$ are denoted by $\bar{v} = [q, \xi]_G$. Using the isomorphism $\alpha_{\mathcal{A}}$, we can consider ℓ as a function defined on $T(Q/G) \oplus \tilde{\mathfrak{g}}$, and we write $\ell(x, \dot{x}, \bar{v})$ to emphasize the dependence of ℓ on $(x, \dot{x}) \in T(Q/G)$ and $\bar{v} \in \tilde{\mathfrak{g}}$. However one should keep in mind that $x, \dot{x}$, and $\bar{v}$ cannot be considered as being independent variables unless $T(Q/G)$ and $\tilde{\mathfrak{g}}$ are trivial bundles.

We now formulate the Lagrangian reduction theorem.

Theorem 4.1 *The following conditions are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_0}^{t_1} L(q, \dot{q}) dt = 0,$$

holds, for variations $\delta q(t)$ vanishing at the endpoints.

ii *The curve $q(t)$ satisfies the Euler-Lagrange equations for L on TQ.*

iii *The reduced variational principle*

$$\delta \int_{t_0}^{t_1} \ell(x, \dot{x}, \bar{v}) dt = 0$$

holds, for variations $\delta x \oplus \delta^{\mathcal{A}} \bar{v}$ of the curve $x(t) \oplus \bar{v}(t)$, where $\delta^{\mathcal{A}} \bar{v}$ has the form

$$\delta^{\mathcal{A}} \bar{v} = \frac{D}{Dt} \bar{\eta} - [\bar{v}, \bar{\eta}] + \tilde{\mathcal{B}}(\delta x, \dot{x}), \tag{4.1}$$

with the boundary conditions $\delta x(t_i) = 0$ and $\bar{\eta}(t_i) = 0$, for $i = 0, 1$.

iv *The following **vertical** and **horizontal Lagrange-Poincaré equations**, hold:*

$$\begin{cases} \dfrac{D}{Dt} \dfrac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v}) = -\operatorname{ad}^*_{\bar{v}} \dfrac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v}) \\ \dfrac{\nabla \ell}{\partial x}(x, \dot{x}, \bar{v}) - \dfrac{\nabla}{dt} \dfrac{\partial \ell}{\partial \dot{x}}(x, \dot{x}, \bar{v}) = \left\langle \dfrac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v}), \mathbf{i}_{\dot{x}} \tilde{\mathcal{B}}(x) \right\rangle . \end{cases} \tag{4.2}$$

We now comment on the various expressions appearing in parts **ii** and **iii**. In the expression (4.1), D/Dt denotes the covariant time derivative of the curve $\bar{\eta}(t) \in \tilde{\mathfrak{g}}$ associated to the principal connection $\mathcal{A}$, that is, for $\bar{\eta}(t) = [q(t), \xi(t)]_G$, we have

$$\frac{D}{Dt}\big[q(t), \xi(t)\big]_G = \Big[q(t),\, \dot{\xi}(t) + [\mathcal{A}(q(t), \dot{q}(t)), \xi(t)]\Big]_G.$$

The bracket $[\bar{v}, \bar{\eta}]$ denotes the Lie bracket induced by $\mathfrak{g}$ on each fiber of $\tilde{\mathfrak{g}}$. The two-form $\tilde{\mathcal{B}} \in \Omega^2(Q/G, \tilde{\mathfrak{g}})$ is the curvature on the base Q/G induced by the curvature form $\mathcal{B} = \mathbf{d}\mathcal{A} + [\mathcal{A}, \mathcal{A}] \in \Omega^2(Q, \mathfrak{g})$ of $\mathcal{A}$. Notice that for the formulation of the Lagrange-Poincaré equations, the introduction of an arbitrary connection ∇ on the manifold Q/G is needed. For simplicity a torsion free connection is chosen. The partial derivatives

$$\frac{\partial \ell}{\partial \dot{x}}(x, \dot{x}, \bar{v}) \in T^*_x(Q/G) \quad \text{and} \quad \frac{\partial \ell}{\partial \bar{v}}(x, \dot{x}, \bar{v}) \in \tilde{\mathfrak{g}}^*_x$$

are the usual fiber derivatives of ℓ in the vector bundles $T(Q/G)$ and $\tilde{\mathfrak{g}}$, and

$$\frac{\nabla \ell}{\partial x}(x, \dot{x}, \bar{v}) \in T^*_x(Q/G)$$

is the partial covariant derivative of ℓ relative to the given connection ∇ on Q/G and to the principal connection $\mathcal{A}$ on Q. We refer to [5] for details and proofs regarding the Lagrange-Poincaré equations. Of course, there is an analogue result on the Hamiltonian side. Given a G invariant Hamiltonian H on T^*Q, Poisson reduction yields the so called *Hamilton-Poincaré equations* on the vector bundle $T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*$, see [4]. Given a Hamiltonian $\mathsf{h} = \mathsf{h}(x, \pi, \bar{\mu}) : T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*$, the Hamilton-Poincaré equations are

$$\begin{cases} \dfrac{D}{Dt}\bar{\mu} = -\operatorname{ad}^*_{\frac{\partial \mathsf{h}}{\partial \bar{\mu}}} \bar{\mu} \\ \dot{x} = \dfrac{\partial \mathsf{h}}{\partial \pi} \\ \dfrac{\nabla}{dt}\pi = -\dfrac{\nabla \mathsf{h}}{\partial x} - \left\langle \bar{\mu}, \mathbf{i}_{\frac{\partial \mathsf{h}}{\partial \pi}} \tilde{\mathcal{B}}(x) \right\rangle, \end{cases} \tag{4.3}$$

where $\partial \mathsf{h}/\partial \pi$ and $\partial \mathsf{h}/\partial \bar{\mu}$ are fiber derivatives, $\nabla \mathsf{h}/\partial x$ is the partial derivative defined in terms of the affine connection ∇ on Q/G, see equations (16)-(19) in [4].

4.2 Lagrangian reduction on the coset bundle $G \to G/G_{n_0}$

In this Section we specialize the situation described in §4.1 to consider the case of a Lie group configuration manifold $Q = G$, which is acted on by its isotropy subgroup G_{n_0}, where n_0 is a fixed point on order parameter manifold M. By the quotient $TG/G_{n_0} \simeq T(G/G_{n_0}) \oplus \widetilde{\mathfrak{g}_{n_0}}$ (or, on the Hamiltonian side: $T^*G/G_{n_0} \simeq T^*(G/G_{n_0}) \oplus \widetilde{\mathfrak{g}_{n_0}}^*$), this description provides the Lagrange-Poincaré (or Hamilton-Poincaré) approach to symmetry breaking, thereby extending the new phase space $T^*(G/G_{n_0})$ to include a charge-like variable taking values in the dual isotropy subalgebra $\mathfrak{g}^*_{n_0}$. This process provides a rigorous geometric framework that naturally explains the emergence of the order parameter space G/G_{n_0} as the effective configuration manifold, starting from an

original system defined on the broken symmetry group G. This is the situation always appearing in physical applications (particularly in condensed matter physics) and its geometric formulation apparently differs from the Euler-Poincaré approach, which does not treat G/G_{n_0} as the effective configuration space.

In what follows, we suppose that the isotropy subgroup $G_{n_0} \subset G$ is a Lie group, with $\dim G_{n_0} \geq 1$. Since the G_{n_0}-action is free and proper, we have the right principal bundle $\pi_{n_0} : G \to G/G_{n_0} \simeq \mathrm{Orb}(n_0), \quad g \mapsto gG_{n_0} \simeq gn_0$. Given a principal connection $\mathcal{A}$ we have the usual vector bundle isomorphism

$$TG/G_{n_0} \to T\,\mathrm{Orb}(n_0) \oplus \widetilde{\mathfrak{g}_{n_0}}, \quad [v_g] \mapsto \big((v_g g^{-1})_M(n), [g, \mathcal{A}(v_g)]_{G_{n_0}}\big) \tag{4.4}$$

over $\mathrm{Orb}(n_0)$, where $n = gn_0$ and $\widetilde{\mathfrak{g}_{n_0}}$ is the adjoint bundle of the isotropy subalgebra $\mathfrak{g}_{n_0}$. Recall from Remark 2.5 that we have the diffeomorphism

$$\bar{i}_{n_0} : TG/G_{n_0} \to \mathfrak{g} \times \mathrm{Orb}(n_0) \subset \mathfrak{g} \times M, \quad [v_g]_{G_{n_0}} \mapsto (v_g g^{-1}, gn_0). \tag{4.5}$$

Therefore, by composing with (4.4) we get the vector bundle isomorphism

$$\mathfrak{g} \times \mathrm{Orb}(n_0) \to T\,\mathrm{Orb}(n_0) \oplus \widetilde{\mathfrak{g}_{n_0}}, \quad (\xi, n) \mapsto \big(\xi_M(n), [g, \mathcal{A}(\xi g)]_{G_{n_0}}\big) \tag{4.6}$$

over $\mathrm{Orb}(n_0)$, where $g \in G$ is such that $gn_0 = n$.

As a concrete example of a principal connection, we consider the mechanical connection associated to a Ad-invariant inner product γ on $\mathfrak{g}$. The associated Riemannian metric on G is bi-invariant and is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Recall that we have (see [31])

$$\mathcal{A}(v_g) = \mathbb{I}(g)^{-1}(\mathbf{J}(v_g)).$$

where the *locked inertia tensor* $\mathbb{I}(g) : \mathfrak{g}_{n_0} \to \mathfrak{g}_{n_0}^*$ is given by

$$\langle \mathbb{I}(g)\eta, \zeta \rangle := \langle\!\langle \eta_G(g), \zeta_G(g) \rangle\!\rangle = \langle\!\langle g\eta, g\zeta \rangle\!\rangle = \gamma|_{\mathfrak{g}_{n_0}}(\eta, \zeta), \quad \text{with} \quad \xi, \eta \in \mathfrak{g}_{n_0}.$$

Therefore the explicit expression of the locked inertia tensor is $\mathbb{I}(g)\eta = \gamma|_{\mathfrak{g}_{n_0}}(\eta, \_)$, where $\gamma|_{\mathfrak{g}_{n_0}}$ denotes the restriction of the inner product γ on $\mathfrak{g}$ to $\mathfrak{g}_{n_0} \subset \mathfrak{g}$. Note that when $\xi \in \mathfrak{g}$ then $\mathbb{I}(g)^{-1}\left(\gamma(\xi, \_)|_{\mathfrak{g}_{n_0}}\right) = \mathbb{P}_{n_0}(\xi)$, where $\mathbb{P}_{n_0} : \mathfrak{g} \to \mathfrak{g}_{n_0}$ is the orthogonal projector associated to γ and $\gamma(\xi, \_)|_{\mathfrak{g}_{n_0}}$ denotes the restriction to $\mathfrak{g}_{n_0}$ of the linear form $\gamma(\xi, \_) \in \mathfrak{g}^*$. Indeed, one has

$$\mathbb{I}(g)\,\mathbb{P}_{n_0}(\xi) = \gamma|_{\mathfrak{g}_{n_0}}(\mathbb{P}_{n_0}(\xi), \_) = \gamma(\xi, \_)|_{\mathfrak{g}_{n_0}}$$

The map $\mathbf{J} : TG \to \mathfrak{g}_{n_0}^*$ is given by

$$\langle \mathbf{J}(v_g), \zeta \rangle := \langle\!\langle v_g, \zeta_G(g) \rangle\!\rangle = \langle\!\langle v_g, g\zeta \rangle\!\rangle = \gamma(g^{-1}v_g, \zeta)\big|_{\mathfrak{g}_{n_0}}, \quad \text{for all} \quad \zeta \in \mathfrak{g}_{n_0}.$$

Therefore $\mathbf{J}(v_g) = \gamma(g^{-1}v_g, \_)|_{\mathfrak{g}_{n_0}}$ and

$$\mathcal{A}(v_g) = \mathbb{I}(g)^{-1}(\mathbf{J}(v_g)) = \mathbb{I}(g)^{-1}\Big(\gamma(g^{-1}v_g, \_)\big|_{\mathfrak{g}_{n_0}}\Big) = \mathbb{P}_{n_0}(g^{-1}v_g)$$

In this case, the vector bundle isomorphism (4.4) reads

$$TG/G_{n_0} \to T\,\mathrm{Orb}(n_0) \oplus \widetilde{\mathfrak{g}_{n_0}}, \quad [v_g] \mapsto \big((v_g g^{-1})_M(n), [g, \mathbb{P}_{n_0}(g^{-1}v_g)]_{G_{n_0}}\big) \tag{4.7}$$

and the diffeomorphism (4.6) is

$$\mathfrak{g} \times \mathrm{Orb}(n_0) \to T\,\mathrm{Orb}(n_0) \oplus \widetilde{\mathfrak{g}_{n_0}}, \quad (\xi, n) \mapsto \big(\xi_M(n), [g, \mathbb{P}_{n_0}(\mathrm{Ad}_{g^{-1}}\,\xi)]_{G_{n_0}}\big) \tag{4.8}$$

In order to compute the curvature of the mechanical connection, we use the formula $\mathcal{B} = \mathbf{d}\mathcal{A} + [\mathcal{A}, \mathcal{A}]$. We have

$$\mathbf{d}\mathcal{A}(u_g, v_g) = \mathbf{d}(\mathcal{A}(Y))X - \mathbf{d}(\mathcal{A}(X))Y - \mathcal{A}([X, Y]),$$

where $X, Y \in \mathfrak{X}(G)$ are two vector fields extending u_g, v_g. Using the left-invariant vector fields $X(g) = g\xi$ and $Y(g) = g\eta$, where $\xi, \eta \in \mathfrak{g}$, we have

$$\mathbf{d}\mathcal{A}(u_g, v_g) = -\mathcal{A}([X, Y](g)) = -\mathcal{A}(g[\xi, \eta]) = -\mathbb{P}_{n_0}([\xi, \eta])$$

thus, we get

$$\mathcal{B}(u_g, v_g) = [\mathbb{P}_{n_0}(\xi), \mathbb{P}_{n_0}(\eta)] - \mathbb{P}_{n_0}([\xi, \eta]), \quad \xi = g^{-1}u_g, \;\; \eta = g^{-1}v_g. \tag{4.9}$$

Suppose that we have chosen a fixed reference point $n_0 \in M$. Using Lagrangian reduction (Theorem 4.1) for the G_{n_0}-invariant Lagrangian $L_{n_0} : TG \to \mathbb{R}$ and the reduced Lagrangian $\ell_{n_0} : T\,\mathrm{Orb}(n_0) \oplus \widetilde{\mathfrak{g}_{n_0}}$, we obtain the Lagrange-Poincaré equations (4.2) in terms of $(n, \dot{n}) \in T\,\mathrm{Orb}(n_0)$'.

Remark 4.2 If the Lagrangian $L_{n_0} : TG \to \mathbb{R}$ is hyperregular, one can obtain the Hamiltonian description for the corresponding Hamiltonian $H_{n_0} : T^*G \to \mathbb{R}$. The reduction process is analogue and one ends up with the *Hamilton-Poincaré equations* on the vector bundle $T^*\,\mathrm{Orb}(n_0) \oplus \widetilde{\mathfrak{g}_{n_0}}^*$, see (4.3) with reduced Hamiltonian h_{n_0} for the general theory. We will refer to ℓ_{n_0} as the *Lagrange-Poincaré (LP) Lagrangian* and to h_{n_0} as the *Hamilton-Poincaré (HP) Hamiltonian*.

4.3 Two equivalent approaches for symmetry breaking

At this stage, it is useful to discuss the relation between the Euler-Poincaré and Lagrange-Poincaré descriptions associated to a G-invariant Lagrangian $L : TG \times M \to \mathbb{R}$. The equivalence of the two approaches arises as follows. Fix a reference point $n_0 \in M$ and consider the induced Lagrangians $L_{n_0} : TG \to \mathbb{R}$, $l : \mathfrak{g} \times M \to \mathbb{R}$, and $\ell_{n_0} : T\,\mathrm{Orb}(n_0) \oplus_{\mathrm{Orb}(n_0)} \widetilde{\mathfrak{g}_{n_0}} \to \mathbb{R}$. Then the following are equivalent:

- $g \in G$ is a solution of the *Euler-Lagrange equations* for L_{n_0}.
- $\xi := \dot{g}g^{-1} \in \mathfrak{g}$ and $n := gn_0 \in M$ are solution of the *Euler-Poincaré equations* (2.6) for l.
- $n := gn_0 \in \mathrm{Orb}(n_0)$ and $\bar{\xi} := [g, \mathcal{A}(\dot{g})]_{G_{n_0}} \in \widetilde{\mathfrak{g}_{n_0}}$ are solutions of the *Lagrange-Poincaré equations* (4.2) for ℓ_{n_0}.

Remark 4.3 (The case of transitive actions) In the case where $G = \mathcal{O}$ is an order parameter group acting transitively on the order parameter space M, we have $\mathrm{Orb}(n_0) = M$, thus the reduced Lagrangian is defined on the vector bundle $TM \oplus_M \widetilde{\mathfrak{p}}$, where $\mathfrak{p}$ is the Lie algebra of $\mathcal{P}$. In particular the diffeomorphisms (4.4), (4.5), (4.6) become

$$\begin{aligned}
&T\mathcal{O}/\mathcal{P} \to TM \oplus \widetilde{\mathfrak{p}}, \quad [v_\chi]_\mathcal{P} \mapsto \big((v_\chi\chi^{-1})_M(n), [\chi, \mathcal{A}(v_\chi)]_\mathcal{P}\big)\\
&\bar{i}_{n_0} : T\mathcal{O}/\mathcal{P} \to \mathfrak{o} \times M, \quad [v_\chi]_\mathcal{P} \mapsto \big(v_\chi\chi^{-1}, \chi n_0\big)\\
&\mathfrak{o} \times M \to TM \oplus \widetilde{\mathfrak{p}}, \quad (\nu, n) \mapsto (\nu_M(n), [\chi, \mathcal{A}(\nu\chi)]_\mathcal{P})\,.
\end{aligned}$$

4.4 Lagrange-Poincaré formulation of uniaxial nematics

We now treat the particular case of *nematics*. Here the order parameter space is the projective plane $M = \mathbb{R}P^2$. Recall that $\mathbb{R}P^2$ is the non-orientable two-dimensional manifold given by the quotient of the two-sphere by the antipodal relation. An element $\mathbf{n} \in \mathbb{R}P^2$ is an equivalence class $\mathbf{n} = [\mathbf{v}]$, where $\mathbf{v} \in S^2$ is a unit vector. The broken symmetry is the group of rotations $SO(3)$ acting on the directors on the left by $\mathbf{n} \mapsto \chi\mathbf{n} := [\chi\mathbf{v}]$. The isotropy group of a fixed direction $\mathbf{n}_0$ is the infinite dihedral group $\mathcal{P} = D_\infty$ generated by the rotations around the axis $\mathbf{n}_0$ and rotation by π around an orthogonal axis. When the z-direction $[(0,0,1)]$ is chosen for the reference axis, then the isotropy group D_∞ consists of matrices of the form

$$\mathcal{M} = \begin{pmatrix} \overline{\mathcal{M}} & 0 \\ 0 & \det(\overline{\mathcal{M}}) \end{pmatrix} \in D_\infty\,, \qquad \text{with } \ \overline{\mathcal{M}} \in O(2). \tag{4.10}$$

The formula above shows how the group D_∞ is actually isomorphic to $O(2)$, which is the group usually appearing in the condensed matter literature.

Remark 4.4 The group D_∞ should not be confused with $D_{\infty,h}$ which contains also $-I_3$ (i.e. minus the identity matrix) and is not a subgroup of $SO(3)$ but of $O(3)$. Note that D_∞ is not Abelian. For example the π-rotation $\mathrm{diag}(1,-1,-1)$ does not commute with rotations around the vertical axis.

The Lie algebra $\mathfrak{d}_\infty$ of D_∞ is given by matrices of the form

$$\begin{pmatrix} 0 & -\sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma \in \mathbb{R}.$$

Using the hat map, it can be identified with the subspace $\mathrm{Span}(0,0,1)$ of $\mathbb{R}^3$ on which the adjoint action acts by matrix multiplication. By identifying $\mathrm{Span}(0,0,1)$ with the real line, and using the notations of (4.10) we obtain the adjoint action

$$\mathrm{Ad}_{\mathcal{M}}\,\sigma = \det(\overline{\mathcal{M}})\,\sigma$$

(recall that $\det(\overline{\mathcal{M}}) = \pm 1$). The identification $SO(3)/D_\infty \simeq \mathbb{R}P^2$ is given by

$$[\chi] \in SO(3)/D_\infty \mapsto \mathbf{n} = [(\chi_{13}, \chi_{23}, \chi_{33})] \in \mathbb{R}P^2 \simeq S^2/\mathbb{Z}_2.$$

Giving a reference point $\mathbf{n}_0 \in \mathbb{R}P^2$ and a Lagrangian $L_{\mathbf{n}_0} : TSO(3) \to \mathbb{R}$ describing the nematic particle, one can easily obtain the Euler-Poincaré formulation associated to the Lagrangian $l : \mathfrak{so}(3) \times \mathbb{R}P^2 \to \mathbb{R}$. In order to obtain the Lagrange-Poincaré equations for $\ell_{\mathbf{n}_0} : TSO(3)/D_\infty \to \mathbb{R}$, one needs to use the adjoint bundle. As we will see, it will be more comfortable to work with the sphere S^2 instead of the projective plane. The adjoint bundle of the right principal bundle $SO(3) \to \mathbb{R}P^2$ is the quotient space

$$\widetilde{\mathfrak{d}_\infty} = \left(SO(3) \times \mathfrak{d}_\infty\right)/D_\infty = \left(SO(3) \times \mathbb{R}\right)/D_\infty$$

relative to the right action of $\mathcal{M} \in D_\infty$ on $(\chi, r) \in SO(3) \times \mathbb{R}$ given by

$$(\chi, r) \mapsto (\chi\mathcal{M}, \det(\overline{\mathcal{M}})r)\,.$$

Note that an element in the fiber $\left(\widetilde{\mathfrak{d}_\infty}\right)_{\mathbf{n}}$ of the vector bundle $\widetilde{\mathfrak{d}_\infty} \to \mathbb{R}P^2$ reads $[\chi, r]$, where $\chi \in SO(3)$ is such that $\mathbf{n} = \chi \mathbf{n}_0$, that is, $[\chi] = \mathbf{n}$. The fact that $\mathcal{M}$ also acts on r is due to the fact that D_∞ is not Abelian.

In order to simplify the approach and to work with more explicit formulas, we replace the order parameter space $\mathbb{R}P^2$ with the two sphere S^2. The breaking symmetry group $SO(2)$ is now Abelian. The reduced Lagrangian is now defined on $\mathfrak{so}(3) \times S^2$ but one has to recall that it is invariant under a change of sign for the variable in S^2. In this case, the isotropy group is $SO(2)$, and the adjoint bundle $\widetilde{\mathfrak{so}(2)} = SO(3) \times_{SO(2)} \mathbb{R}$ is a trivial bundle, since $SO(2)$ is Abelian and thus the adjoint action on its Lie algebra $\mathfrak{so}(2) \simeq \mathbb{R}$ is trivial.

We now describe the mechanical connection associated to the Ad-invariant inner product $\gamma(\xi, \eta) = \frac{1}{2}\mathrm{trace}(\xi^T \eta)$ on $\mathfrak{so}(3)$. Recall that the Lie algebra $\mathfrak{so}(3)$ and $\mathfrak{so}(2)$ are identified with $\mathbb{R}^3$ and $\mathbb{R}(0,0,1)$, via the hat map. On $\mathbb{R}^3$ the inner product γ is the standard inner product and the projection $\mathbb{P} : \mathfrak{so}(3) \simeq \mathbb{R}^3 \to \mathfrak{so}(2) \simeq \mathbb{R}$ is simply given by taking the third component. The mechanical connection is thus given by

$$\mathcal{A}(v_\chi) = \mathbb{P}(\chi^{-1} v_\chi) = (\chi^{-1} v_\chi)_3. \tag{4.11}$$

According to formula (4.9), the curvature is

$$\mathcal{B}(u_\chi, v_\chi) = 0 - (\boldsymbol{\nu} \times \boldsymbol{\kappa})_3 = -\nu_1 \kappa_2 + \nu_2 \kappa_1, \qquad \boldsymbol{\nu} = \chi^{-1} u_\chi, \boldsymbol{\kappa} = \chi^{-1} v_\chi.$$

In order to compute the reduced curvature on S^2 we first note that the tangent map to the projection $\pi_{\mathbf{n}_0} : SO(3) \to S^2$, $\chi \mapsto \chi \mathbf{n_0}$ reads

$$T\pi_{\mathbf{n}_0} : TSO(3) \to TS^2, \quad \hat{\boldsymbol{\nu}}\chi \mapsto \boldsymbol{\nu} \times \chi \mathbf{n}_0 = \boldsymbol{\nu} \times \mathbf{n}.$$

Note that if $\dot{\mathbf{n}} \in T_{\mathbf{n}} S^2$ is given, then we have $T\pi_{\mathbf{n}_0}\left(\widehat{\mathbf{n} \times \dot{\mathbf{n}}}\,\chi\right) = \dot{\mathbf{n}}$, where the 'hat' map denotes the usual isomorphism $\mathbb{R}^3 \simeq \mathfrak{so}(3)$. Therefore, the reduced curvature is

$$\begin{aligned}
\tilde{\mathcal{B}}_{\mathbf{n}}(\dot{\mathbf{n}}, \dot{\mathbf{m}}) &= \mathcal{B}\left(\widehat{\mathbf{n} \times \dot{\mathbf{n}}}\,\chi\,,\, \widehat{\mathbf{n} \times \dot{\mathbf{m}}}\,\chi\right) = -\left(\chi^{-1}(\mathbf{n} \times \dot{\mathbf{n}}) \times \chi^{-1}(\mathbf{n} \times \dot{\mathbf{m}})\right)_3 \\
&= -\left((\mathbf{n}_0 \times \chi^{-1}\dot{\mathbf{n}}) \times (\mathbf{n}_0 \times \chi^{-1}\dot{\mathbf{m}})\right)_3 = -\left(\chi^T \dot{\mathbf{n}} \times \chi^T \dot{\mathbf{m}}\right)_3 \\
&= -\left(\chi^T(\dot{\mathbf{n}} \times \dot{\mathbf{m}})\right)_3 = -\mathbf{n} \cdot (\dot{\mathbf{n}} \times \dot{\mathbf{m}}).
\end{aligned} \tag{4.12}$$

In the last equality we used the formula $(\chi^T \boldsymbol{\nu})_3 = \chi_{i3} \boldsymbol{\nu}_i = \mathbf{n} \cdot \boldsymbol{\nu}$, valid since $\chi \in SO(3)$ is such that $\chi \mathbf{n_0} = \mathbf{n}$, where $\mathbf{n}_0 = (0,0,1)$. Up to a sign the curvature is given by the volume of the polytope generated by the vectors $\mathbf{n}$, $\dot{\mathbf{n}}$ and $\dot{\mathbf{m}}$.

Using the mechanical connection, and the fact that the adjoint bundle is trivial, the diffeomorphism (4.8) reads

$$\mathfrak{so}(3) \times S^2 \to TS^2 \times \mathbb{R}, \qquad (\boldsymbol{\nu}, \mathbf{n}) \mapsto (\mathbf{n}, \boldsymbol{\nu} \times \mathbf{n}, \mathbf{n} \cdot \boldsymbol{\nu}) = (\mathbf{n}, \dot{\mathbf{n}}, r). \tag{4.13}$$

Indeed, the infinitesimal generator associated to $\boldsymbol{\nu} \in \mathfrak{so}(3) \simeq \mathbb{R}^3$ reads as $\boldsymbol{\nu}_{S^2}(\mathbf{n}) = \boldsymbol{\nu} \times \mathbf{n} \in T_{\mathbf{n}} S^2$ and the second component of (4.8) is

$$\mathbb{P}(\mathrm{Ad}_{\chi^{-1}} \boldsymbol{\nu}) = (\chi^{-1} \boldsymbol{\nu})_3 = (\chi^T \boldsymbol{\nu})_3 = \chi_{i3} \boldsymbol{\nu}_i = \mathbf{n} \cdot \boldsymbol{\nu},$$

In terms of equivalence classes $[v_\chi] \in TSO(3)/SO(2)$, the variables $\mathbf{n}$ and $\boldsymbol{\nu}$ are given by $\mathbf{n} = \chi \mathbf{n}_0$, and $\boldsymbol{\nu} = v_\chi \chi^{-1}$.

We now obtain the formula for the inverse of the vector bundle map (4.13). Denoting $r := \mathbf{n}\cdot\boldsymbol{\nu}$ and $\dot{\mathbf{n}} := \boldsymbol{\nu}\times\mathbf{n}$ we have $\mathbf{n}\times(\boldsymbol{\nu}\times\mathbf{n}) = (\mathbf{n}\cdot\mathbf{n})\boldsymbol{\nu} - \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\nu}) = \boldsymbol{\nu} - r\mathbf{n}$, and

$$\boldsymbol{\nu} = \mathbf{n}\times(\boldsymbol{\nu}\times\mathbf{n}) + r\mathbf{n} = \mathbf{n}\times\dot{\mathbf{n}} + r\mathbf{n}. \tag{4.14}$$

This proves that the inverse of the vector bundle map (4.13) is

$$(\mathbf{n},\dot{\mathbf{n}},r)\in TS^2\times\mathbb{R}\mapsto(\mathbf{n}\times\dot{\mathbf{n}}+r\mathbf{n},\mathbf{n}) = (\boldsymbol{\nu},\mathbf{n})\in\mathfrak{so}(3)\times S^2.$$

Therefore, the Lagrangian l in (3.3) and the LP Lagrangian $\ell_{\mathbf{n}_0}$ (now simply denoted by ℓ, recall that $\mathbf{n}_0 = (0,0,1)$), are related by

$$l(\boldsymbol{\nu},\mathbf{n}) = \ell(\mathbf{n},\boldsymbol{\nu}\times\mathbf{n},\mathbf{n}\cdot\boldsymbol{\nu}),\quad \ell(\mathbf{n},\dot{\mathbf{n}},r) = l(\mathbf{n}\times\dot{\mathbf{n}}+r\mathbf{n},\mathbf{n}). \tag{4.15}$$

We now compute the Lagrange-Poincaré equations associated to the reduced Lagrangian $\ell = \ell(\mathbf{n},\dot{\mathbf{n}},r) : TS^2\times\mathbb{R}\to\mathbb{R}$. Since the group $SO(2)$ is Abelian, the Lie bracket is zero, the covariant derivative coincides with usual derivative. Using formula (4.12) for the reduced curvature of the mechanical connection, the Lagrange-Poincaré equations (4.2) read

$$\frac{d}{dt}\frac{\partial\ell}{\partial r} = 0,\quad \frac{\nabla}{dt}\frac{\partial\ell}{\partial\dot{\mathbf{n}}} - \frac{\nabla\ell}{\partial\mathbf{n}} = \frac{\partial\ell}{\partial r}\mathbf{n}\times\dot{\mathbf{n}}, \tag{4.16}$$

where the curvature term is evidently given by $\mathbf{n}\times\dot{\mathbf{n}} = -\mathbf{i}_{\dot{\mathbf{n}}}\mathcal{B}_{\mathbf{n}}$, and ∇/dt, $\nabla\ell/\partial\mathbf{n}$ are associated to the Levi-Civita connection on S^2 induced by the inner product on $\mathbb{R}^3$. The explicit form of the Lagrangian $\ell_{\mathbf{n}_0}$ can be written immediately upon using (4.15) and recalling equation (3.7). We obtain the LP Lagrangian

$$\begin{aligned}\ell(\mathbf{n},\dot{\mathbf{n}},r) = l(\mathbf{n}\times\dot{\mathbf{n}}+r\mathbf{n},\mathbf{n}) &= \frac{1}{2}\,j\,|\mathbf{n}\times\dot{\mathbf{n}}+r\mathbf{n}|^2 - \phi(\mathbf{n})\\ &= \frac{1}{2}\,j\,|\mathbf{n}\times\dot{\mathbf{n}}|^2 + \frac{1}{2}\,j\,r^2 - \phi(\mathbf{n})\\ &= \frac{1}{2}\,j\,|\dot{\mathbf{n}}|^2 + \frac{1}{2}\,j\,r^2 - \phi(\mathbf{n})\,.\end{aligned} \tag{4.17}$$

where the potential term may be expressed again by the quadratic expression $\phi(\mathbf{n}) = \lambda\,|\mathbf{n}\cdot\mathbf{k}|^2$, or alternatively by $\phi(\mathbf{n}) = \lambda\,|\mathbf{n}\times\mathbf{k}|^2$, for a fixed number λ and an unsigned unit vector $\mathbf{k}$. Upon inserting the expression (4.17) in the Lagrange-Poincaré equations (4.16) we get

$$\dot{r} = 0,\quad j\frac{\nabla}{dt}\dot{\mathbf{n}} + \frac{\nabla\phi}{\partial\mathbf{n}} = j\,r\,\mathbf{n}\times\dot{\mathbf{n}}.$$

Note that ϕ does not depend on $\dot{\mathbf{n}}$, so $\nabla\phi/\partial\mathbf{n}$ coincides with the gradient of the map $\phi : S^2\to\mathbb{R}$ with respect to the Riemannian metric induced by $\mathbb{R}^3$ on S^2. In order to recover the usual form of the equations, we will interpret $\mathbf{n}$ as a curve in $\mathbb{R}^3$ and ϕ as a map defined on $\mathbb{R}^3$. In this case we get the equation

$$j\,\ddot{\mathbf{n}} - 2q\mathbf{n} + \nabla\phi(\mathbf{n}) = j\,r\,\dot{\mathbf{n}}\times\mathbf{n}\qquad\text{with}\qquad 2q := \mathbf{n}\cdot(j\,\ddot{\mathbf{n}} + \nabla\phi(\mathbf{n}))$$

and where $\nabla\phi$ denotes the gradient of ϕ view as a map on $\mathbb{R}^3$. This equation evidently reduces to the celebrated Ericksen-Leslie equation for the case $r = 0$ (cf. [6]).

Remark 4.5 (Physical nature of the variable r) The conserved variable $r = \boldsymbol{\nu}_0 \cdot \mathbf{n}_0$ is evidently the projection of the angular velocity of the nematic molecule on its director. Thus, when this quantity is non-zero, it encodes the effect of rotations of the molecule about the director. Because of the particular form of the Lagrangian, this effect is taken into account only by the curvature term on the right hand side of the Euler-Lagrange equation. An analogous situation holds for the heavy top dynamics (see remark 4.6 below). However, for nematic molecules, such rotations about the director are irrelevant, due to the rod-like nature of nematics. Thus, the convention $r = 0$ producing Ericksen-Leslie dynamics is the most natural in this case. On the other hand, this is not true for the dynamics of the heavy top, which is a rigid body (with fixed point) of arbitrary shape.

For the Hamiltonian side, one needs the dual vector bundle map of (4.15) over S^2. We have

$$(\boldsymbol{\mu}, \mathbf{n}) \in \mathfrak{so}(3)^* \times S^2 \mapsto (\mathbf{n}, \boldsymbol{\pi}, w) = (\mathbf{n}, \boldsymbol{\mu} \times \mathbf{n}, \boldsymbol{\mu} \cdot \mathbf{n}) \in T^*S^2 \times \mathbb{R}^*, \tag{4.18}$$

whose inverse is given by

$$(\mathbf{n}, \boldsymbol{\pi}, w) \in T^*S^2 \times \mathbb{R}^* \mapsto (\mathbf{n} \times \boldsymbol{\pi} + w\mathbf{n}, \mathbf{n}) = (\boldsymbol{\mu}, \mathbf{n}) \in \mathfrak{so}(3)^* \times S^2.$$

Therefore, the Hamiltonian h in (3.5) produces the corresponding HP Hamiltonian by $\mathsf{h}(\mathbf{n}, \boldsymbol{\pi}, w) = h(\mathbf{n} \times \boldsymbol{\pi} + w\mathbf{n}, \mathbf{n})$. The nematic particle Hamiltonian reads as

$$\mathsf{h}(\mathbf{n}, \boldsymbol{\pi}, w) = \frac{1}{2j}|\boldsymbol{\pi}|^2 + \frac{1}{2j}\, w^2 + \phi(\mathbf{n}) \tag{4.19}$$

which evidently differs from the Lie-Poisson Hamiltonian

$$h(\boldsymbol{\mu}, \mathbf{n}) = \frac{1}{2j}\, |\boldsymbol{\mu}|^2 + \phi(\mathbf{n})$$

only by the constant factor w, without changing the content in physical information. In the case of (4.19), the quadratic nature of the Hamiltonian produces a Kaluza-Klein construction analogue to that underlying Wong's equations in Yang-Mills theory. In this setting, substitution of the Hamiltonian (4.19) in the the Hamilton-Poincaré equations (4.3) yields

$$\dot{w} = 0, \quad j\dot{\mathbf{n}} = \boldsymbol{\pi}, \quad \frac{\nabla}{dt}\boldsymbol{\pi} = -\frac{\nabla \phi}{\partial \mathbf{n}} - \frac{w}{j}\boldsymbol{\pi} \times \mathbf{n},$$

so that the usual Ericksen-Leslie equation is recovered for $w = 0$.

Remark 4.6 (Comparison with the heavy top) It is evident that the above reduction can be performed equivalently for the heavy top dynamics by expanding the quotient $TSO(3)/SO(2) \simeq TS^2 \oplus \widetilde{\mathbb{R}}$, which gives analogous equations to those above. Although the Lagrangian (and the Hamiltonian) keep the same form as the single nematic particle, the main difference resides in the form of the potential, whose explicit expression is now $\phi(\boldsymbol{\Gamma}) = \lambda\, \boldsymbol{\Gamma} \cdot \mathbf{k}$, where we use the notation of §1.2. This expression is evidently $SO(2)$-invariant, rather than D_∞-invariant, consistently with the broken symmetry of the heavy top system. Indeed, in the heavy top case, the dynamical variable $\mathbf{n}$ belongs to the sphere $S^2 \simeq SO(3)/SO(2)$ rather than to the projective plane $\mathbb{R}P^2 \simeq SO(3)/D_\infty$, thereby reflecting the different nature of the two breaking symmetry subgroups $SO(2)$ and D_∞. Moreover, we remark that the Ericksen-Leslie equation also holds for the heavy top dynamics provided the potential is now $\phi(\boldsymbol{\Gamma}) = \lambda\, \boldsymbol{\Gamma} \cdot \mathbf{k}$. However, in this case, setting $r = 0$ has no physical motivation.

Remark 4.7 (Lagrange-Poincaré approach for moving nematic particles) From a physical point of view, one should also take into account the translational motion of nematic particles in the physical space Q, thereby extending the LP Lagrangian to $\ell : TQ \times (TS^2 \times \mathbb{R}) \to \mathbb{R}$. This process has the only effect of producing an extra Euler-Lagrange equation on physical space, without changing any of the present geometric construction. The final equations are then

$$\frac{d}{dt}\frac{\partial \ell}{\partial \dot{q}} - \frac{\partial \ell}{\partial q} = 0\,, \qquad \frac{\nabla}{dt}\frac{\partial \ell}{\partial \dot{\mathbf{n}}} - \frac{\nabla \ell}{\partial \mathbf{n}} = \frac{\partial \ell}{\partial r}\mathbf{n} \times \dot{\mathbf{n}}\,, \qquad \frac{d}{dt}\frac{\partial \ell}{\partial r} = 0.$$

with the Lagrangian

$$\ell(q, \dot{q}, \mathbf{n}, \dot{\mathbf{n}}, r) = \frac{1}{2}\,\|\dot{q}\|^2 + \frac{1}{2}\,j\,|\dot{\mathbf{n}}|^2 - \Phi(\mathbf{n}) + \frac{1}{2}\,j\,r^2$$

Remark 4.8 (Biaxial nematics) Recall that the case of biaxial nematic particles involves the order parameter space $SO(3)/D_2$, where D_2 is the finite dihedral group. This discrete symmetry cannot be considered under the preceding Lagrange-Poincaré approach, since the latter is defined only for order parameter spaces $\mathcal{O}/\mathcal{P}$, involving an isotropy subgroup $\mathcal{P} \subset \mathcal{O}$, such that $\dim(\mathcal{P}) \geq 1$. When the isotropy subgroup $\mathcal{P}$ is discrete, then the situation requires more care in order to take into account the trivial nature of its Lie algebra $\mathfrak{p} = \{0\}$. The broken symmetry group $\mathcal{O}$ becomes a principal bundle $\mathcal{O} \to \mathcal{O}/\mathcal{P}$ with discrete fiber and the reduction process in this case remains unknown.

4.5 Summary for uniaxial nematics

This section gives an overview that summarizes the different approaches that have been carried out so far for uniaxial nematic particles. The starting point is the the unreduced $SO(3)$-invariant Lagrangian $L : TSO(3) \times S^2 \to \mathbb{R}$ that describes the dynamics of a single molecule. Then, we fix a reference direction $\mathbf{n}_0$ and consider the induced Lagrangians

$$\begin{aligned} L_{\mathbf{n}_0} &: TSO(3) \longrightarrow \mathbb{R}, \quad L_{\mathbf{n}_0} = L_{\mathbf{n}_0}(v_\chi) \\ l &: \mathfrak{so}(3) \times S^2 \longrightarrow \mathbb{R}, \quad l = l(\boldsymbol{\nu}, \mathbf{n}) \\ \ell_{\mathbf{n}_0} &: TS^2 \times \mathbb{R} \longrightarrow \mathbb{R}, \quad \ell_{\mathbf{n}_0} = \ell_{\mathbf{n}_0}(\mathbf{n}, \dot{\mathbf{n}}, r), \end{aligned}$$

where we choose the reference configuration $\mathbf{n}_0 = (0, 0, 1)$ for simplicity. Given a curve $\chi \in SO(3)$ and the vectors $\mathbf{n} = \chi \mathbf{n}_0 \in S^2$ and $\boldsymbol{\nu} = \dot{\chi}\chi^{-1} \in \mathfrak{so}(3)$, the following are equivalent:

i The curve χ is a solution of the *Euler-Lagrange equations* for $L_{\mathbf{n}_0}$.

ii The curves $\boldsymbol{\nu}$ and $\mathbf{n}$ are solutions of the *Euler-Poincaré equations* for l:

$$\frac{d}{dt}\frac{\delta l}{\delta \boldsymbol{\nu}} + \frac{\delta l}{\delta \boldsymbol{\nu}} \times \boldsymbol{\nu} = \mathbf{n} \times \frac{\delta l}{\delta \mathbf{n}} \qquad \dot{\mathbf{n}} = \boldsymbol{\nu} \times \mathbf{n}. \tag{4.20}$$

iii The curves $\mathbf{n}$ and r are solutions of the *Lagrange-Poincaré equations* for $\ell_{\mathbf{n}_0}$:

$$\frac{d}{dt}\frac{\partial \ell_{\mathbf{n}_0}}{\partial r} = 0, \quad \frac{d}{dt}\frac{\partial \ell_{\mathbf{n}_0}}{\partial \dot{\mathbf{n}}} - \frac{\partial \ell_{\mathbf{n}_0}}{\partial \mathbf{n}} = \frac{\partial \ell_{\mathbf{n}_0}}{\partial r}\mathbf{n} \times \dot{\mathbf{n}}. \tag{4.21}$$

To obtain the equations (4.20) from the Euler-Poincaré equations (2.6), we used the formulas $\mathrm{ad}^*_{\boldsymbol{\nu}}\boldsymbol{\kappa} = -\boldsymbol{\nu}\times\boldsymbol{\kappa}$ and the expression of the momentum map

$$\mathbf{J} : T^*S^2 \to \mathfrak{so}(3)^*, \quad \mathbf{J}(\mathbf{n},\boldsymbol{\pi}) = \mathbf{n}\times\boldsymbol{\pi}.$$

for the cotangent lifted action of $SO(3)$ on T^*S^2. The conservation law (2.7) for nematics reads

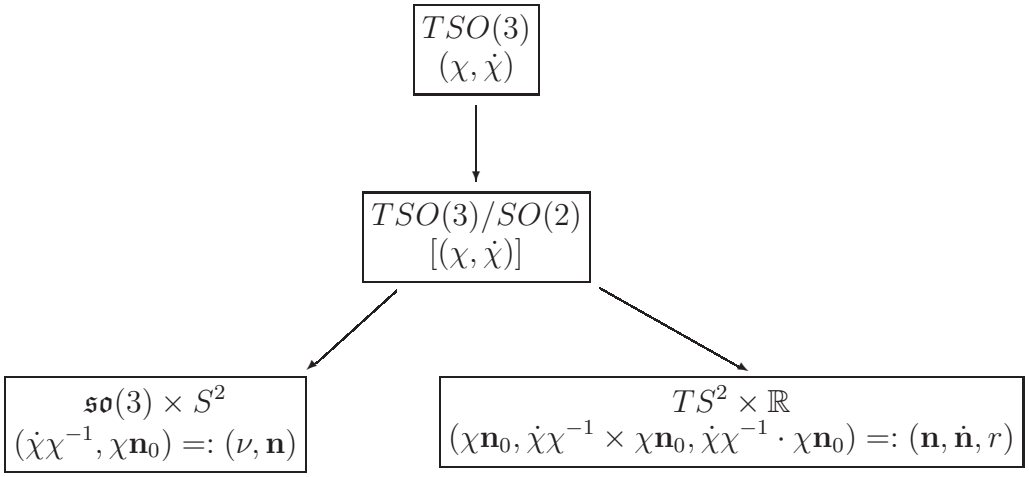
$$\frac{\partial}{\partial t}\left(\chi^{-1}\frac{\delta l}{\delta \boldsymbol{\nu}}\right) = \mathbf{n}_0 \times \left(\chi^{-1}\frac{\delta l}{\delta \mathbf{n}}\right).$$

On can pass from the Euler-Poincaré (4.20) to the Lagrange-Poincaré equations (4.21), using the relations:

$$\boldsymbol{\nu} = \mathbf{n}\times\dot{\mathbf{n}} + r\mathbf{n}$$

$$r = \mathbf{n}\cdot\boldsymbol{\nu}, \quad \dot{\mathbf{n}} = \boldsymbol{\nu}\times\mathbf{n}.$$

The link between the Lagrangian variables $(\chi,\dot{\chi})$ and the reduced variables $(\nu,\mathbf{n})$ and $(\mathbf{n},\dot{\mathbf{n}},r)$ is illustrated in the following diagram.



Remark 4.9 (unit sphere vs. unit vectors) Instead of the sphere $M = S^2$, one can start with the vector space $M = \mathbb{R}^3$ on which $SO(3)$ acts by matrix multiplication. This is the point of view adopted in [13]. In this case, if one chooses the initial condition $\mathbf{n}_0 \in S^2$, then we still have $\mathbf{n}(t) = \chi(t)\mathbf{n}_0 \in S^2$, thus these approaches are equivalent.

Remark 4.10 (Directors and the real projective plane) Recall that the parameter space for nematics is the projective plane $\mathbb{R}P^2$ and not the sphere S^2. However, the results **i** – **iv** are still true when one works with the projective plane, that is, when $\mathbf{n}$ is interpreted as a director instead as a unit vector. The only change concerns the Lagrange-Poincaré equations which are less explicit in the case of the projective plane. This is the reason why we use S^2 instead of $\mathbb{R}P^2$ above.

Remark 4.11 (Euler-Poincaré equations via Lagrangian reduction) From our discussions we realize that the Euler-Poincaré and Lagrange-Poincaré equations arise from two different approaches in reduction theory. However it is possible to obtain the Euler-Poincaré equations by standard Lagrangian reduction, provided one suitably enlarges the physical configuration space. We address the reader to [3, 11] for the case of order parameter vector spaces. This picture produces the so called Clebsch constrained variational principle [15].

5 Hydrodynamics of nematic liquid crystals

While the previous discussions have focused on the geometric dynamics of a single particle with broken symmetry, this section develops the same ideas in the more physical situation of a fluid system of particles with micro-structure, such as nematic particles. Thus, this section provides the link between the geometric treatment previously applied at the microscopic single-particle level and the macroscopic models usually adopted for liquid crystal dynamics. We shall see how the same ideas apply to continuum media without substantial modifications. The extension to nematic fluid dynamics requires considering the diffeomorphism group, i.e. the particle relabeling group well known in the Lagrangian picture of fluid dynamics.

We begin by recalling our notation: $\mathcal{O}$ be the order parameter group of a certain particle with broken symmetry, and denote by M the order parameter space, on which $\mathcal{O}$ acts transitively, with isotropy group $\mathcal{P}$. As we have seen before, passing from one particle to a system of N particles simply consists in replacing the group $\mathcal{O}$ by the direct product $\mathcal{O}^N$ acting on the cartesian product M^N. When a continuum of particles with broken symmetry is considered, one needs to consider as symmetry group the group $G := \mathcal{F}(\mathcal{D}, \mathcal{O})$ of smooth maps form the physical space $\mathcal{D}$ to the order parameter group $\mathcal{O}$ of the single particle. We still denote by $\chi : \mathcal{D} \to \mathcal{O}$ these maps. The order parameter space is the manifold of maps $\mathcal{M} := \mathcal{F}(\mathcal{D}, M)$ on which G acts by the naturally induced pointwise action.

In order to describe the hydrodynamic of such systems of particles (called complex fluids), we need to include the group of diffeomorphisms in the symmetry group. For simplicity, we suppose that the fluid is incompressible, since the compressible case would require only a slight modification. Thus we need to consider the group $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$ of volume preserving diffeomorphisms of $\mathcal{D}$, relative to a fixed volume form μ on $\mathcal{D}$. It will be convenient to fix a Riemannian metric g on $\mathcal{D}$ and to choose μ as the volume form associated to the metric.

5.1 Euler-Poincaré formulation

In the Euler-Poincaré framework, the dynamics of complex fluids is obtained by considering the semidirect product $G = \mathrm{Diff}_{\mathrm{vol}}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ acting on advected variables by linear or affine representation as shown in [13]. Here the advected variable is the order parameter function $n \in \mathcal{F}(\mathcal{D}, M)$, on which G acts by the left action

$$n \mapsto (\eta, \chi) n := (\chi n) \circ \eta^{-1}, \quad (\eta, \chi) \in G. \tag{5.1}$$

Recall that χn denotes the action of $\mathcal{F}(\mathcal{D}, \mathcal{O})$ on $\mathcal{F}(\mathcal{D}, M)$ naturally induced by the action of $\mathcal{O}$ on M. One easily checks that (5.1) defines an action of the semidirect product $G = \mathrm{Diff}_{\mathrm{vol}}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$. Using the expression

$$\mathrm{ad}^*_{(u,\nu)}(m, \kappa) = \left(\pounds_u m + \kappa \cdot \nabla \nu, \pounds_u \kappa + \mathrm{ad}^*_\nu \kappa \right),$$

for the infinitesimal coadjoint action, we obtain from (2.6) the equations

$$\begin{cases} \dfrac{\partial}{\partial t} \dfrac{\delta l}{\delta u} + \pounds_u \dfrac{\delta l}{\delta u} + \dfrac{\delta l}{\delta \nu} \cdot \nabla \nu = -\dfrac{\delta l}{\delta n} \cdot \nabla n - \nabla p, \qquad \mathrm{div}\, u = 0 \\ \dfrac{\partial}{\partial t} \dfrac{\delta l}{\delta \nu} + \pounds_u \dfrac{\delta l}{\delta \nu} + \mathrm{ad}^*_\nu \dfrac{\delta l}{\delta \nu} = \mathbf{J} \circ \dfrac{\delta l}{\delta n}, \end{cases} \tag{5.2}$$

together with the kinematic equation

$$\dot{n} + u\cdot\nabla n = \nu_M \circ n.$$

The Lagrangian $L_{n_0} : T\big(\operatorname{Diff}_{\mathrm{vol}}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D},\mathcal{O})\big) \to \mathbb{R}$ is usually of the form

$$L_{n_0}(\eta,\dot\eta,\chi,\dot\chi) = \frac{1}{2}\int_{\mathcal{D}} \|\dot\eta\|^2\mu + \frac{1}{2}\int_{\mathcal{D}} j|\dot\chi|^2\mu - \int_{\mathcal{D}} F\left((\chi n_0)\circ\eta^{-1}, \nabla((\chi n_0)\circ\eta^{-1})\right)\mu,$$

where the norm of $\dot\eta$ is given by the Riemannian metric g, the norm of $\dot\chi$ is associated to a right-invariant Riemannian metric on $\mathcal{O}$, and F is the free energy. One easily checks that this expression is invariant under the right action of $(\eta,\chi) \in G = \operatorname{Diff}_{\mathrm{vol}}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D},\mathcal{O})$ given by cotangent lift on TG and acting on n_0 as $n_0 \mapsto \chi^{-1}(n_0\circ\eta)$. The reduced Euler-Poincaré Lagrangian recovers the expression

$$l(u,\nu,n) = \frac{1}{2}\int_{\mathcal{D}} \|u\|^2\mu + \frac{1}{2}\int_{\mathcal{D}} j|\nu|^2\mu - \int_{\mathcal{D}} F(n,\nabla n)\mu, \tag{5.3}$$

where F is the free energy.

5.1.1 Euler-Poincaré fluid equations for nematic molecules

The case of nematic liquid crystals can be treated as above, choosing $\mathcal{O} = SO(3)$, $M = \mathbb{R}P^2$. In this case the free energy F is usually given by the Oseen-Zöcher-Frank expression

$$F(\mathbf{n},\nabla\mathbf{n}) = \frac{1}{2}K_1(\operatorname{div}\mathbf{n})^2 + \frac{1}{2}K_2(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + \frac{1}{2}K_3|\mathbf{n}\times\operatorname{curl}\mathbf{n}|^2, \tag{5.4}$$

where the constants K_1, K_2, K_3 are respectively associated to the three principal distinct director axis deformations in nematics, namely, splay, twist, and bend. At this point, the system (5.2) produces the equations

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\partial_i\left(\nabla\mathbf{n}^T\cdot\dfrac{\partial F}{\partial\mathbf{n}_{,i}}\right) - \nabla p, \qquad \operatorname{div} u = 0 \\[2ex] j\left(\dot{\boldsymbol{\nu}} + u\cdot\nabla\boldsymbol{\nu}\right) = \mathbf{h}\times\mathbf{n}, \qquad \mathbf{h} := -\dfrac{\delta l}{\delta\mathbf{n}} = \dfrac{\partial F}{\partial\mathbf{n}} - \partial_i\left(\dfrac{\partial F}{\partial\mathbf{n}_{,i}}\right) \end{cases} \tag{5.5}$$

together with the kinematic equation

$$\dot{\mathbf{n}} + u\cdot\nabla\mathbf{n} = \boldsymbol{\nu}\times\mathbf{n}.$$

The Ericksen-Leslie fluid equations follow easily from the above relations, under the assumption that the initial condition verifies $\boldsymbol{\nu}_0\cdot\mathbf{n}_0 = 0$ [13]. A direct computation using (2.9) shows that the associated Poisson bracket reads

$$\begin{aligned}\{f,g\}(m,\boldsymbol{\mu},\mathbf{n}) = &\left\langle m, \left[\frac{\delta f}{\delta m}, \frac{\delta g}{\delta m}\right]\right\rangle + \left\langle \boldsymbol{\mu}, \left(\frac{\delta f}{\delta\boldsymbol{\mu}}\times\frac{\delta g}{\delta\boldsymbol{\mu}} + \nabla\frac{\delta f}{\delta\boldsymbol{\mu}}\cdot\frac{\delta g}{\delta m} - \nabla\frac{\delta g}{\delta\boldsymbol{\mu}}\cdot\frac{\delta f}{\delta m}\right)\right\rangle \\ &+ \left\langle \mathbf{n}\times\frac{\delta f}{\delta\boldsymbol{\mu}} + \nabla\mathbf{n}\cdot\frac{\delta f}{\delta m}, \frac{\delta g}{\delta\mathbf{n}}\right\rangle - \left\langle \mathbf{n}\times\frac{\delta g}{\delta\boldsymbol{\mu}} + \nabla\mathbf{n}\cdot\frac{\delta g}{\delta m}, \frac{\delta f}{\delta\mathbf{n}}\right\rangle,\end{aligned} \tag{5.6}$$

where $m = \delta l/\delta u$ is the fluid momentum and the brackets $\langle\,,\rangle$ denote L^2 duality. The first two terms is the Lie-Poisson bracket, associated to the semidirect product $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))$. Of course, this bracket is consistent with that of the single particle, see (3.6).

From the discussion above, it is easy to generalize the Euler-Poincaré fluid equations for uniaxial nematics to the case of biaxial liquid crystals. Indeed, upon choosing $\mathcal{O} = SO(3)$ and $M = SO(3)/D_2$, these equations can be written down directly as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta u} + \pounds_u \dfrac{\delta l}{\delta u} + \dfrac{\delta l}{\delta \boldsymbol{\nu}}\cdot\nabla\boldsymbol{\nu} = -\displaystyle\sum_{l=1}^{2}\nabla\mathbf{n}_l\cdot\frac{\delta l}{\delta \mathbf{n}_l} - \nabla p, \qquad \operatorname{div} u = 0 \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \boldsymbol{\nu}} + \operatorname{div}\left(\dfrac{\delta l}{\delta \boldsymbol{\nu}}\,u\right) - \boldsymbol{\nu}\times\dfrac{\delta l}{\delta \boldsymbol{\nu}} = \displaystyle\sum_{l=1}^{2}\mathbf{n}_l\times\frac{\delta l}{\delta \mathbf{n}_l}\,, \end{cases}$$

together with the kinematic equation

$$\dot{\mathbf{n}}_l + u\cdot\nabla\mathbf{n}_l = \mathbf{n}_l\times\boldsymbol{\nu}\,, \qquad l = 1, 2.$$

On the Hamiltonian side, the corresponding Poisson bracket reads

$$\begin{aligned}\{f,g\}(m,\boldsymbol{\mu},\mathbf{n}_1,\mathbf{n}_2) = &\left\langle m, \left[\frac{\delta f}{\delta m}, \frac{\delta g}{\delta m}\right]\right\rangle + \left\langle \boldsymbol{\mu}, \left(\frac{\delta f}{\delta \boldsymbol{\mu}}\times\frac{\delta g}{\delta \boldsymbol{\mu}} + \nabla\frac{\delta f}{\delta \boldsymbol{\mu}}\cdot\frac{\delta g}{\delta m} - \nabla\frac{\delta g}{\delta \boldsymbol{\mu}}\cdot\frac{\delta f}{\delta m}\right)\right\rangle \\ &+ \sum_{l=1}^{2}\left\langle \mathbf{n}_l\times\frac{\delta f}{\delta \boldsymbol{\mu}} + \nabla\mathbf{n}_l\cdot\frac{\delta f}{\delta m}, \frac{\delta g}{\delta \mathbf{n}_l}\right\rangle - \left\langle \mathbf{n}_l\times\frac{\delta g}{\delta \boldsymbol{\mu}} + \nabla\mathbf{n}_l\cdot\frac{\delta g}{\delta m}, \frac{\delta f}{\delta \mathbf{n}_l}\right\rangle.\end{aligned}$$

The next section extends the Euler-Poincaré approach for the alignment tensor dynamics of a single nematic molecule to the case of nematic liquid crystals.

5.1.2 Alignment tensor dynamics in nematic liquid crystals

It is well known that fluid equations are obtained from the single particle dynamics by taking statistical averages with respect to some probability distribution function. The case of liquid crystals is not an exception to this procedure and one is usually interested in the dynamics of the averaged alignment tensor $\bar{\mathsf{Q}}$, where the 'bar' symbol denotes an appropriate averaging of the single particle alignment tensor Q. In order to formulate the Euler-Poincaré equations for liquid crystals in terms of the averaged quantity $\bar{\mathsf{Q}}$, one proceeds by simply replacing the order parameter space $M = \mathbb{R}P^2$ with the space of symmetric matrices $\mathrm{Sym}(3)$. The Euler-Poincaré reduction process is performed on the unreduced Lagrangian $L_{\mathsf{Q}_0} : T\big(\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},SO(3))\big)\to\mathbb{R}$ with respect to the action

$$(\eta,\chi)\mathsf{Q} = \left(\chi\mathsf{Q}\chi^{-1}\right)\circ\eta^{-1},$$

where evidently $\mathsf{Q}\in\mathcal{F}(\mathcal{D},\mathrm{Sym}(3))$ and we have suppressed the averaging notation. Thus, if one specializes the general equations (5.2) to the case under consideration, one obtains

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta u} + \pounds_u \dfrac{\delta l}{\delta u} + \dfrac{\delta l}{\delta \boldsymbol{\nu}}\cdot\nabla\boldsymbol{\nu} = -\nabla\mathsf{Q}^{ij}\dfrac{\delta l}{\delta \mathsf{Q}^{ij}} - \nabla p, \qquad \operatorname{div} u = 0 \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \boldsymbol{\nu}} + \operatorname{div}\left(\dfrac{\delta l}{\delta \boldsymbol{\nu}}\,u\right) - \boldsymbol{\nu}\times\dfrac{\delta l}{\delta \boldsymbol{\nu}} = -\overrightarrow{\left[\mathsf{Q}, \dfrac{\delta l}{\delta \mathsf{Q}}\right]}, \qquad \dot{\mathsf{Q}} + u\cdot\nabla\mathsf{Q} = [\hat{\boldsymbol{\nu}},\mathsf{Q}]\,, \end{cases} \tag{5.7}$$

or more explicitly, for a Lagrangian of the form (5.3) with n replaced by Q,

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\partial_k \left(\dfrac{\partial F}{\partial \mathsf{Q}^{ij}_{,k}} \right) \nabla \mathsf{Q}^{ij} - \nabla p, & \operatorname{div} u = 0 \\ j\left(\dot{\boldsymbol{\nu}} + u\cdot\nabla\boldsymbol{\nu}\right) = \overrightarrow{[\mathsf{Q},\mathsf{H}]}, & \dot{\mathsf{Q}} + u\cdot\nabla\mathsf{Q} = [\hat{\boldsymbol{\nu}},\mathsf{Q}]\,, \end{cases} \tag{5.8}$$

where the molecular field H is given by

$$\mathsf{H} = -\frac{\delta l}{\delta \mathsf{Q}} = \frac{\partial F}{\partial \mathsf{Q}} - \partial_k \left(\frac{\partial F}{\partial \mathsf{Q}_{,k}} \right)$$

with the notation $\overrightarrow{A}_i = \varepsilon_{ijk} A_{jk}$ and the summation convention over repeated tensor indexes. The free energy $F(\mathsf{Q}, \nabla\mathsf{Q})$ appearing in the reduced Lagrangian $l(u, \boldsymbol{\nu}, \mathsf{Q})$ is usually given by the Landau-de Gennes free energy in the form [7]

$$F(\mathsf{Q}, \nabla\mathsf{Q}) = \kappa_1 \left\|\nabla\mathsf{Q}\right\|^2 + \kappa_2 \left\|\nabla\cdot\mathsf{Q}\right\|^2 + \kappa_3 \operatorname{Tr}(\mathsf{Q}\,\nabla\times\mathsf{Q}) + a_{21}\operatorname{Tr}(\mathsf{Q}^2) + a_{31}\operatorname{Tr}(\mathsf{Q}^3)$$

where the tensor norms are given by total contraction of the indexes, i.e. $\left\|\nabla\mathsf{Q}\right\|^2 = \sum \left(\partial_k \mathsf{Q}^{ij}\right)^2$ and $\left\|\nabla\cdot\mathsf{Q}\right\|^2 = \sum \left(\partial_i \mathsf{Q}^{ij} \partial_k \mathsf{Q}^{kj}\right)$, while $(\nabla\times)_{ij} = \epsilon_{ijk}\partial_k$ is considered as a matrix operator. Higher order expansions in the alignment tensor are also possible.

On the Hamiltonian side, the corresponding Poisson bracket reads

$$\begin{aligned} \{f,g\}(m,\boldsymbol{\mu},\mathsf{Q}) = &\left\langle m, \left[\frac{\delta f}{\delta m}, \frac{\delta g}{\delta m}\right]\right\rangle + \left\langle \boldsymbol{\mu}, \left(\frac{\delta f}{\delta \boldsymbol{\mu}} \times \frac{\delta g}{\delta \boldsymbol{\mu}} + \nabla\frac{\delta f}{\delta \boldsymbol{\mu}}\cdot\frac{\delta g}{\delta m} - \nabla\frac{\delta g}{\delta \boldsymbol{\mu}}\cdot\frac{\delta f}{\delta m}\right)\right\rangle \\ &+ \operatorname{Tr}\left(\left[\mathsf{Q}, \frac{\delta f}{\delta \hat{\boldsymbol{\mu}}}\right]\frac{\delta g}{\delta \mathsf{Q}} + \left(\frac{\delta f}{\delta m}\cdot\nabla\right)\mathsf{Q}\,\frac{\delta g}{\delta \mathsf{Q}}\right) - \operatorname{Tr}\left(\left[\mathsf{Q}, \frac{\delta g}{\delta \hat{\boldsymbol{\mu}}}\right]\frac{\delta f}{\delta \mathsf{Q}} + \left(\frac{\delta g}{\delta m}\cdot\nabla\right)\mathsf{Q}\,\frac{\delta f}{\delta \mathsf{Q}}\right) \end{aligned}$$

and it is easy to recognize that the same relations also hold for biaxial liquid crystals.

Remark 5.1 (Compressible fluid flows) In the Euler-Poincaré setting, the generalization to compressible fluid flows is straightforward. Indeed, it is sufficient to consider the whole diffeomorphism group $\operatorname{Diff}(\mathcal{D})$ and to enlarge the order parameter space by replacing $\mathcal{F}(\mathcal{D}, M)$ with the product $\mathcal{F}(\mathcal{D}, M) \times \operatorname{Den}(\mathcal{D})$, where $\operatorname{Den}(\mathcal{D})$ denotes the space of densities on $\mathcal{D}$. Then one repeats the construction above, upon considering the following action of $G = \operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ on $\mathcal{F}(\mathcal{D}, M) \times \operatorname{Den}(\mathcal{D})$

$$(\eta, \chi)\cdot(\rho, \mathbf{n}) = \left((\rho\circ\eta) J(\eta), \chi^{-1}(\mathbf{n}\circ\eta)\right)\,.$$

where $J(\eta)$ is the Jacobian of η with respect to a volume form on $\mathcal{D}$. In this sense, compressibility is by itself another example of symmetry breaking in infinite dimensions (cf. [22]). Indeed, even in the case of ordinary isotropic compressible fluids, the full Diff-symmetry of the unreduced Lagrangian $L_{\rho_0} : T\operatorname{Diff}(\mathcal{D}) \to \mathbb{R}$ is broken by the presence of the density variable $\rho_0 \in \operatorname{Den}(\mathcal{D})$, so that the only symmetry which is left is given by the infinite-dimensional isotropy subgroup $\operatorname{Diff}_{\rho_0}(\mathcal{D})$. Upon applying the Euler-Poincaré approach, one performs the reduction in (3.1): $T\operatorname{Diff}(\mathcal{D})/\operatorname{Diff}_{\rho_0}(\mathcal{D}) \simeq \mathfrak{X}(\mathcal{D}) \times \operatorname{Den}(\mathcal{D})$ because of the transitivity of the action (see e.g. [25]). In principle, it is also possible to apply Lagrangian reduction and write the Lagrange-Poincaré equations on the reduced space $T\operatorname{Diff}(\mathcal{D})/\operatorname{Diff}_{\rho_0}(\mathcal{D}) \simeq T\operatorname{Den}(\mathcal{D}) \oplus \widetilde{\mathfrak{X}_{\mathrm{vol}}(\mathcal{D})}$. Although the next discussion is devoted to Lagrangian reduction, we shall consider again the case of incompressible nematic liquid crystals.

5.2 Lagrange-Poincaré formulation of uniaxial nematic liquid crystals

The Lagrange-Poincaré approach to continuum nematic media produces another variant of the fluid equations for liquid crystals. One starts with the same Lagrangian as before, namely

$$L_{\mathbf{n}_0}(\eta,\dot\eta,\chi,\dot\chi)=\frac{1}{2}\int_{\mathcal{D}}\|\dot\eta\|^2\mu+\frac{1}{2}\int_{\mathcal{D}}j\,|\dot\chi|^2\,\mu-\int_{\mathcal{D}}F\left((\chi\mathbf{n}_0)\circ\eta^{-1},\nabla((\chi\mathbf{n}_0)\circ\eta^{-1})\right)\mu.$$

However we will now reduce the dynamics in two steps. The first step is a Lagrange-Poincaré reduction involving the internal variables. The second step involves the diffeomorphism group and is done by using the metamorphosis approach.

5.2.1 First reduction stage: Lagrange-Poincaré approach

As a first step of reduction, one applies the Lagrange-Poincaré approach for $L_{\mathbf{n}_0}$ according to the vector bundle isomorphism

$$T\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times T\mathcal{F}(\mathcal{D},SO(3))/\mathcal{F}(\mathcal{D},SO(2))\ \to\ T\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times\left(T\mathcal{F}(\mathcal{D},S^2)\oplus\mathcal{F}(\mathcal{D})\right)$$

over $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times\mathcal{F}(\mathcal{D},S^2)$. By a slight abuse of notation, we replaced $\mathrm{Orb}(n_0)\subset\mathcal{F}(\mathcal{D},S^2)$ with $\mathcal{F}(\mathcal{D},S^2)$ itself. Also, we have replaced the isotropy group $\mathcal{F}(\mathcal{D},D_\infty)$ with the simpler choice $\mathcal{F}(\mathcal{D},SO(2))$, in analogy with the preceding finite-dimensional treatment. In this case, upon fixing $\mathbf{n}_0=(0,0,1)$, the Lagrange-Poincaré reduction with respect to the mechanical connection yields the LP Lagrangian $\ell(\eta,\dot\eta,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}},\bar r)$ on the reduced space

$$T\,\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times\left(T\mathcal{F}(\mathcal{D},S^2)\oplus\mathcal{F}(\mathcal{D})\right)\ni(\eta,\dot\eta,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}},\bar r)\,.$$

The LP Lagrangian is

$$\ell(\eta,\dot\eta,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}},\bar r)=\frac{1}{2}\int_{\mathcal{D}}\|\dot\eta\|^2\mu+\frac{j}{2}\int_{\mathcal{D}}|\dot{\bar{\mathbf{n}}}|^2\mu+\frac{j}{2}\int_{\mathcal{D}}\bar r^2\mu-\int_{\mathcal{D}}F(\bar{\mathbf{n}}\circ\eta^{-1},\nabla(\bar{\mathbf{n}}\circ\eta^{-1})\mu.$$

At this point, one recalls that the quantity $\bar r\in\mathcal{F}(\mathcal{D})$ in the above Lagrangian is a constant parameter, due to the form of the Lagrangian. Thus, it is allowed to set $\bar r\equiv 0$ for convenience, so that the rotational nematic dynamics is regulated by the ordinary Euler-Lagrange equations on $\mathcal{F}(\mathcal{D},S^2)$. Therefore, one obtains the Lagrangian $\ell':T\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times T\mathcal{F}(\mathcal{D},S^2)\to\mathbb{R}$ given by

$$\begin{aligned}\ell'(\eta,\dot\eta,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}}):&=\ell(\eta,\dot\eta,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}},0)\\&=\frac{1}{2}\int_{\mathcal{D}}\|\dot\eta\|^2\mu+\frac{j}{2}\int_{\mathcal{D}}|\dot{\bar{\mathbf{n}}}|^2\mu-\int_{\mathcal{D}}F(\bar{\mathbf{n}}\circ\eta^{-1},\nabla(\bar{\mathbf{n}}\circ\eta^{-1}))\mu\end{aligned}$$

where one can verify directly the invariance of the free energy F under the right action

$$(\eta,\dot\eta,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}})\cdot\varphi=(\eta\circ\varphi,\dot\eta\circ\varphi,\bar{\mathbf{n}}\circ\varphi,\dot{\bar{\mathbf{n}}}\circ\varphi)$$

by simply observing that

$$F\big((\bar{\mathbf{n}}\circ\varphi)\circ(\eta\circ\varphi)^{-1},\nabla\big((\bar{\mathbf{n}}\circ\varphi)\circ(\eta\circ\varphi)^{-1}\big)\big)=F(\bar{\mathbf{n}}\circ\eta^{-1},\nabla(\bar{\mathbf{n}}\circ\eta^{-1})).$$

We now endow the manifold $\mathcal{F}(\mathcal{D},S^2)$ with the Levi-Civita connection associated to the Riemannian metric given by integration over $\mathcal{D}$ of the natural Riemannian on $S^2\subset\mathbb{R}^3$. Upon denoting

by ∇/dt the covariant derivative with respect to the Levi-Civita connection of S^2, one writes the explicit form of the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \ell'}{\partial \dot\eta} - \frac{\partial \ell'}{\partial \eta} = 0, \quad \frac{\nabla}{dt}\frac{\partial \ell'}{\partial \dot{\bar{\mathbf{n}}}} - \frac{\nabla \ell'}{\partial \bar{\mathbf{n}}} = 0$$

on the tangent bundle of $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, S^2)$. The above equations are found for example in [15], where they are shown to be equivalent to the Ericksen-Leslie fluid's equations. Of course, one can obtain these equations from (5.5), when the initial condition $\mathbf{n}_0$ and $\boldsymbol{\nu}_0$ are orthogonal.

5.2.2 Second reduction stage: metamorphosis approach

The Eulerian form of the fluid equations is found by noting that the Lagrangian ℓ' is invariant under the cotangent-lift of the right action $(\eta, \bar{\mathbf{n}}) \mapsto (\eta \circ \varphi, \bar{\mathbf{n}} \circ \varphi)$ of the diffeomorphism $\varphi \in \mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$. Thus, one performs the reduction process according to the quotient map

$$T\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D}) \times T\mathcal{F}(\mathcal{D}, S^2) \to \mathfrak{X}_{\mathrm{vol}}(\mathcal{D}) \times T\mathcal{F}(\mathcal{D}, S^2)$$

given by $(\eta, \dot\eta, \bar{\mathbf{n}}, \dot{\bar{\mathbf{n}}}) \mapsto (u, \mathbf{n}, D_t\mathbf{n}) := (\dot\eta \circ \eta^{-1}, \bar{\mathbf{n}} \circ \eta^{-1}, \dot{\bar{\mathbf{n}}} \circ \eta^{-1})$ so that the reduced Lagrangian is

$$\ell(u, \mathbf{n}, D_t\mathbf{n}) = \frac{1}{2}\int_{\mathcal{D}} \|u\|^2 \mu + \frac{j}{2}\int_{\mathcal{D}} |D_t\mathbf{n}|^2\mu - \int_{\mathcal{D}} F(\mathbf{n}, \nabla\mathbf{n})\mu.$$

This particular form of Lagrangian reduction is used in [24] to formulate the metamorphosis equations in imaging science. See also [12] for a geometric description of metamorphosis reduction. Note that we have chosen to denote by $D_t\mathbf{n}$ the variable in $T_{\mathbf{n}}\mathcal{F}(\mathcal{D}, S^2)$ since the dynamics will yield the relation $D_t\mathbf{n} = (\partial_t + u\cdot\nabla)\mathbf{n}$. A direct computation shows that the reduced equations are given by

$$\begin{cases} \dot u + u\cdot\nabla u = -\partial_i\left(\nabla\mathbf{n}^T\cdot\dfrac{\partial F}{\partial \mathbf{n}_{,i}}\right) - \nabla p, \qquad \mathrm{div}\, u = 0 \\ j\left(\dfrac{\nabla}{dt} + u\cdot\nabla\right) D_t\mathbf{n} = \mathbf{h}, \qquad \mathbf{h} := -\dfrac{\delta l}{\delta \mathbf{n}} = \dfrac{\partial F}{\partial \mathbf{n}} - \partial_i\left(\dfrac{\partial F}{\partial \mathbf{n}_{,i}}\right). \end{cases} \tag{5.9}$$

These equations recover the Ericksen-Leslie fluid equations.

Remark 5.2 This second reduction step is a standard Lagrangian reduction and not an Euler-Poincaré reduction as described in Section 2.1, since $T\mathcal{F}(\mathcal{D}, S^2)$ is part of the tangent bundle and may not be interpreted as a parameter space.

Remark 5.3 (Metamorphosis for parameter-dependent Lagrangians) In order to account for compressibility in a natural way, one needs to extend the Lagrangian to depend also on the parameter $\rho_0 \in \mathrm{Den}(\mathcal{D})$, so that

$$L_{(\rho_0, \mathbf{n}_0)} : T\mathrm{Diff}(\mathcal{D}) \times T\mathcal{F}(\mathcal{D}, SO(3)) \to \mathbb{R}.$$

Then, after the first Lagrange-Poincaré reduction stage on $\mathbf{n}_0 = (0, 0, 1)$, one obtains the reduced Lagrangian

$$\ell'_{\rho_0} : T\mathrm{Diff}(\mathcal{D}) \times \big(T\mathcal{F}(\mathcal{D}, S^2) \oplus \mathcal{F}(\mathcal{D})\big) \to \mathbb{R}.$$

At this point, after setting $\bar{r}=0$, one performs a metamorphosis reduction stage on the above Lagrangian according to the invariance property

$$\ell'(\eta,\dot{\eta},\rho_0,\bar{\mathbf{n}},\dot{\bar{\mathbf{n}}})=\ell'(\dot{\eta}\circ\eta^{-1},(\rho_0\circ\eta^{-1})J(\eta^{-1}),\bar{\mathbf{n}}\circ\eta^{-1},\dot{\bar{\mathbf{n}}}\circ\eta^{-1})=\ell(u,\rho,\mathbf{n},D_t\mathbf{n})$$

where we have used the same notation as in Remark 5.1, although the above reduction process differs from that described in 5.1. Therefore, the reduced Lagrangian ℓ is defined such as

$$\ell:\mathfrak{X}(\mathcal{D})\times\mathrm{Den}(\mathcal{D})\times T\mathcal{F}(\mathcal{D},S^2)\to\mathbb{R}.$$

The reduction process just described is an example of a metamorphosis reduction for parameter-dependent Lagrangians. Notice that this process can be extended to any type of G-invariant Lagrangian

$$L_{m_0}:TG\times TN\to\mathbb{R}.$$

where N is a manifold, while m_0 is a parameter belonging to another order parameter space M. This method extends the Euler-Poincaré approach presented in §2.1.

5.2.3 Hamilton-Poincaré formulation and its Poisson bracket

Notice that one can Legendre-transform the equations thereby obtaining a Hamiltonian $\mathsf{h}(m,\mathbf{n},\boldsymbol{\pi})$, on $\mathfrak{X}^*_{\mathrm{vol}}(\mathcal{D})\times T^*\mathcal{F}(\mathcal{D},S^2)$. Indeed, upon introducing the fluid momentum and the conjugate director variables

$$m=\frac{\delta\ell}{\delta u}\in\mathfrak{X}^*_{\mathrm{vol}}(\mathcal{D})\quad\text{and}\quad(\mathbf{n},\boldsymbol{\pi})=\left(\mathbf{n},\frac{\delta\ell}{\delta(D_t\mathbf{n})}\right)\in T^*\mathcal{F}(\mathcal{D},S^2)\,,$$

the Hamiltonian functional

$$\mathsf{h}(m,\mathbf{n},\boldsymbol{\pi})=\langle m\,,u\rangle+\langle\boldsymbol{\pi}\,,D_t\mathbf{n}\rangle-\ell(u,\mathbf{n},D_t\mathbf{n})$$

produces the following Poisson bracket via Legendre transformation

$$\begin{aligned}\{f,g\}(m,\mathbf{n},\boldsymbol{\pi})=&\left\langle m,\left[\frac{\delta f}{\delta m},\frac{\delta g}{\delta m}\right]\right\rangle+\{f,\,g\}_{T^*\mathcal{F}(\mathcal{D},S^2)}\\&+\left\langle\frac{\delta f}{\delta(\mathbf{n},\boldsymbol{\pi})}\,,\pounds_{\frac{\delta g}{\delta m}}(\mathbf{n},\boldsymbol{\pi})\right\rangle-\left\langle\frac{\delta g}{\delta(\mathbf{n},\boldsymbol{\pi})}\,,\pounds_{\frac{\delta f}{\delta m}}(\mathbf{n},\boldsymbol{\pi})\right\rangle,\end{aligned}\tag{5.10}$$

where $\{\cdot,\,\cdot\}_{T^*\mathcal{F}(\mathcal{D},S^2)}$ denotes the canonical Poisson bracket on $T^*\mathcal{F}(\mathcal{D},S^2)$, $\pounds$ denotes Lie derivative and the Lie bracket $[\cdot,\,\cdot]$ stands for minus the Jacobi-Lie bracket on vector fields, as usual in fluid mechanics. The same Hamiltonian structure can also be obtained by a two steps reduction from the Hamiltonian $H_{n_0}:T^*\big(\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times\mathcal{F}(\mathcal{D},SO(3))\big)\to\mathbb{R}$ associated to L_{n_0}. In this process, one proceeds analogously by ignoring the conjugate variable $\bar{w}=\delta\ell/\delta\bar{r}$ (momentum associated to $\bar{r}$), which is possible because of the special form of the Hamiltonian.

The Poisson bracket formulation of condensed matter systems (especially liquid crystals) is a rather relevant topic in the physics literature; cf e.g. [9, 38]. Here we emphasize that the Poisson structures (5.6) and (5.10) arise from *two different reductions* of the canonical Hamiltonian structure on $T^*\big(\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})\times\mathcal{F}(\mathcal{D},SO(3))\big)$ and they both produce *the same Ericksen-Leslie equations*.

The Hamiltonian system described by (5.10) is of the general form $h : \mathfrak{g}^* \times P \to \mathbb{R}$, upon choosing $\mathfrak{g} = \mathfrak{X}_{\text{vol}}(\mathcal{D})$ and $P = T^*\mathcal{F}(\mathcal{D}, S^2)$. However, this construction differs from that treated in the first part of this paper and one cannot simply transfer the Lie-Poisson setting discussed previously to this infinite-dimensional case. Indeed, the difference resides in the fact that P carries its own Poisson structure and thus it is a Poisson manifold by itself. Such a construction appears quite often in condensed matter systems, where $P = T^*\mathcal{F}(\mathcal{D}, M)$ [18], and it also emerges in electromagnetic fluid dynamics, where $P = T^*\Omega^1(\mathcal{D})$ is the phase space of Maxwell equations [17]. This type of Hamiltonian systems arising from an unreduced Hamiltonian on $T^*G \times P$ has been extensively studied in [26], where many interesting properties are presented. For example, an interesting consequence of the special type of the bracket (5.10) is that it allows for a Poisson isomorphism that eliminates all the terms in the second line of (5.10). More precisely, given an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$, there exists a Poisson diffeomorphism $(\mu, p) \mapsto (\mu + \mathbf{J}(p), p)$ sending the reduced Poisson structure on $(T^*G \times P)/G = \mathfrak{g}^* \times P$ to the product Poisson structure on $(T^*G/G) \times P = \mathfrak{g}^* \times P$ (cf. Proposition 2.2 in [26]). In the case of uniaxial nematics, this corresponds to introducing a new variable

$$\mathsf{m} := m + \mathbf{J}(\mathbf{n}, \boldsymbol{\pi}) \,,$$

so that the new set of variables $(\mathsf{m}, \mathbf{n}, \boldsymbol{\pi})$ carries the following 'untangled' Poisson bracket:

$$\{f, g\}(\mathsf{m}, \mathbf{n}, \boldsymbol{\pi}) = \left\langle \mathsf{m}, \left[\frac{\delta f}{\delta \mathsf{m}}, \frac{\delta g}{\delta \mathsf{m}}\right] \right\rangle + \{f,\, g\}_{T^*\mathcal{F}(\mathcal{D},S^2)} \tag{5.11}$$

This bracket produces ordinary Hamilton's equations on $T^*\mathcal{F}(\mathcal{D}, S^2)$ as well as a Lie-Poisson equation on $\mathfrak{X}^*_{\text{vol}}(\mathcal{D})$. In the special case of the bracket (5.10) for liquid crystals, one can introduce the following momentum map, associated to the cotangent lifted action of $G = \text{Diff}_{\text{vol}}(\mathcal{D})$ on $P = T^*\mathcal{F}(\mathcal{D}, S^2)$:

$$\mathbf{J}(\mathbf{n}, \boldsymbol{\pi}) = \nabla \mathbf{n}^T \cdot \boldsymbol{\pi} + \nabla \varphi \in \mathfrak{X}_{\text{vol}}(\mathcal{D})^* \tag{5.12}$$

where $(\mathbf{n}, \boldsymbol{\pi}) \in T^*\mathcal{F}(\mathcal{D}, S^2)$ and φ is a variable such that $\text{div}(\mathbf{J}(\mathbf{n}, \boldsymbol{\pi})) = 0$. Then, the new Hamiltonian reads as

$$h\left(\mathsf{m}, \mathbf{n}, \boldsymbol{\pi}\right) = \frac{1}{2}\int_{\mathcal{D}} \left\|\mathsf{m} - \nabla \mathbf{n}^T \cdot \boldsymbol{\pi} - \nabla\varphi\right\|^2 \mu + \frac{1}{2j}\int_{\mathcal{D}} |\boldsymbol{\pi}|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, \nabla \mathbf{n})\mu.$$

This approach has been sometimes referred to as 'untangling' and the Poisson bracket (5.11) is called 'untangled Poisson bracket'. See [16, 17, 18] for examples of how the untangling and entangling processes are used in the physics of charged fluids and superfluids.

5.2.4 The helicity invariant

Another important property of Hamiltonian systems of the general form $h : \mathfrak{g}^* \times P \to \mathbb{R}$ involves Casimir functions. Indeed, from the results in [26], it follows immediately that the Poisson bracket (5.10) allows for an interesting class of Casimir functions. Indeed, given an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ and any Casimir function $C(\mu)$ for the Lie-Poisson bracket on $\mathfrak{g}^*$, the function

$$C(\mu, p) = C(\mu + \mathbf{J}(p)) \tag{5.13}$$

is a Casimir for the reduced Poisson bracket on $(T^*G \times P)/G = \mathfrak{g}^* \times P$ (cf. Corollary 2.3 in [26]). In the case of liquid crystals, the momentum map (5.12) can be used to produce an explicit expression for the helicity of nematic liquid crystals. Indeed, since it is well known that the helicity $\mathscr{H}(m) = \langle \operatorname{curl} m, m\rangle$ is a Casimir for the Lie-Poisson bracket on $\mathfrak{X}_{\mathrm{vol}}(\mathcal{D})^*$, then direct substitution of the momentum map $\mathbf{J}(\mathbf{n}, \boldsymbol{\pi})$ in formula (5.13) yields a Casimir for the Poisson bracket (5.10). This Casimir is explicitly written as the following helicity functional for nematic liquid crystals

$$\mathscr{H}(m, \mathbf{n}, \boldsymbol{\pi}) = \int_{\mathcal{D}} \left(m + \nabla \mathbf{n}^T \cdot \boldsymbol{\pi}\right) \cdot \operatorname{curl}\left(m + \nabla \mathbf{n}^T \cdot \boldsymbol{\pi}\right). \tag{5.14}$$

This helicity invariant can also be written as a Casimir for the Lie-Poisson bracket (5.6). Indeed, one can pull back $\mathscr{H}$ using the mapping $(m, \boldsymbol{\mu}, \mathbf{n}) \mapsto (m, \mathbf{n}, \boldsymbol{\mu} \times \mathbf{n})$, which is suggested by the isomorphism (4.18) (with $w = 0$) holding for the dynamics of the single nematic molecule. The resulting expression for the helicity is then

$$\mathscr{H}(m, \boldsymbol{\mu}, \mathbf{n}) = \int_{\mathcal{D}} \left(m + \boldsymbol{\mu} \cdot (\mathbf{n} \times \nabla \mathbf{n})\right) \cdot \operatorname{curl}\left(m + \boldsymbol{\mu} \cdot (\mathbf{n} \times \nabla \mathbf{n})\right). \tag{5.15}$$

More rigorously, the Casimir property can be verified by restricting the Lie-Poisson construction in (5.6) to the submanifold $\mathscr{P} = \{(m, \boldsymbol{\mu}, \mathbf{n}) \mid \boldsymbol{\mu} \cdot \mathbf{n} = 0\}$, which is analogous to setting $w = 0$ in (4.18) for a single nematic molecule. Then, the following arguments show that $\mathscr{P}$ can be endowed with the Poisson structure (5.6), thereby making it into a Poisson submanifold of $\mathcal{W} = \mathfrak{X}_{\mathrm{vol}}(\mathcal{D})^* \times \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))^* \times \mathcal{F}(\mathcal{D}, S^2)$. Let $(m(t), \boldsymbol{\mu}(t), \mathbf{n}(t))$ be a solution of Hamilton's equation $\dot{f} = \{f, h\}$ on $\mathcal{W}$ relative to an arbitrary Hamiltonian h. Since $\partial_t(\boldsymbol{\mu} \cdot \mathbf{n}) + (u \cdot \nabla)(\boldsymbol{\mu} \cdot \mathbf{n}) = 0$, then we obtain $(\boldsymbol{\mu}(t) \cdot \mathbf{n}(t)) = (\boldsymbol{\mu}(0) \cdot \mathbf{n}(0)) \circ \eta_t^{-1}$, where we have denoted by η_t the flow of u. Consequently, any Hamiltonian vector field on $\mathcal{W}$ restricted to $\mathscr{P}$ is tangent to $\mathscr{P}$ and this shows that $\mathscr{P}$ is a quasi Poisson submanifold. Thus, there is a unique Poisson structure on $\mathscr{P}$ making it into a Poisson submanifold, see Prop 4.1.23 in [36]. It is readily seen that this Poisson structure has the same expression (5.6).

Theorem 5.4 *The expression*

$$\mathscr{H}(m, \boldsymbol{\mu}, \mathbf{n}) = \int_{\mathcal{D}} \left(m + \boldsymbol{\mu} \cdot (\mathbf{n} \times \nabla \mathbf{n})\right) \cdot \operatorname{curl}\left(m + \boldsymbol{\mu} \cdot (\mathbf{n} \times \nabla \mathbf{n})\right). \tag{5.16}$$

is a Casimir function on the Poisson manifold $\mathscr{P} = \{(m, \boldsymbol{\mu}, \mathbf{n}) \mid \boldsymbol{\mu} \cdot \mathbf{n} = 0\}$, *endowed with the bracket* (5.6).

Proof. We shall show that for a solution $(m(t), \boldsymbol{\mu}(t), \mathbf{n}(t))$ of an arbitrary Hamiltonian system on $\mathscr{P}$, we have

$$\left(\frac{\partial}{\partial t} + \pounds_{\frac{\delta h}{\delta m}}\right) \mathcal{C} = -\mathbf{d}p.$$

where we have defined the differential one form $\mathcal{C} := m + \boldsymbol{\mu}\cdot(\mathbf{n}\times\nabla\mathbf{n})$. A direct calculation shows that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\frac{\delta h}{\delta m}}\right)\mathcal{C} =& -\boldsymbol{\mu}\cdot\frac{\delta h}{\delta\boldsymbol{\mu}} + \nabla\mathbf{n}\cdot\frac{\delta h}{\delta\mathbf{n}} - \operatorname{grad} p + \left(\frac{\delta h}{\delta\boldsymbol{\mu}}\times\boldsymbol{\mu} + \frac{\delta h}{\delta\mathbf{n}}\times\mathbf{n}\right)\cdot(\mathbf{n}\times\nabla\mathbf{n}) \\
&+ \boldsymbol{\mu}\cdot\left(\left(\frac{\delta h}{\delta\boldsymbol{\mu}}\times\mathbf{n}\right)\times\nabla\mathbf{n}\right) + \boldsymbol{\mu}\cdot\left(\mathbf{n}\times\nabla\left(\frac{\delta h}{\delta\boldsymbol{\mu}}\times\mathbf{n}\right)\right) \\
=& -\boldsymbol{\mu}\cdot\frac{\delta h}{\delta\boldsymbol{\mu}} + \nabla\mathbf{n}\cdot\frac{\delta h}{\delta\mathbf{n}} - \operatorname{grad} p + \boldsymbol{\mu}\cdot\left(\mathbf{n}\times\mathbf{n}\times\nabla\frac{\delta h}{\delta\boldsymbol{\mu}}\right) - \mathbf{n}\cdot\left(\frac{\delta h}{\delta\mathbf{n}}\times\mathbf{n}\times\nabla\mathbf{n}\right) \\
=& -\boldsymbol{\mu}\cdot\frac{\delta h}{\delta\boldsymbol{\mu}} + \nabla\mathbf{n}\cdot\frac{\delta h}{\delta\mathbf{n}} - \operatorname{grad} p - \boldsymbol{\mu}\cdot\left(\left(\nabla\frac{\delta h}{\delta\boldsymbol{\mu}}\cdot\mathbf{n}\right)\mathbf{n} - \nabla\frac{\delta h}{\delta\boldsymbol{\mu}}\right) \\
&- \mathbf{n}\cdot\left(\left(\nabla\mathbf{n}\cdot\frac{\delta h}{\delta\mathbf{n}}\right)\mathbf{n} - \left(\frac{\delta h}{\delta\mathbf{n}}\cdot\mathbf{n}\right)\nabla\mathbf{n}\right) \\
=& -\mathbf{d}p
\end{aligned}$$

where we have used the Jacobi identity for double cross products, and the properties $|\mathbf{n}|^2 = 1$ and $\boldsymbol{\mu}\cdot\mathbf{n} = \mathbf{n}\cdot\nabla\mathbf{n} = 0$, respectively. Thus, upon writing the helicity as $\mathscr{H} = \int_{\mathcal{D}}\mathcal{C}\wedge\mathbf{d}\mathcal{C}$, one obtains

$$\partial_t\left(\mathcal{C}\wedge\mathbf{d}\mathcal{C}\right) = -\mathbf{d}p\wedge\mathbf{d}\mathcal{C} - \operatorname{div}\left(\left(\mathcal{C}\wedge\mathbf{d}\mathcal{C}\right)\frac{\delta h}{\delta m}\right)\mathrm{d}^3x$$

so that $\mathscr{H} = \int_{\mathcal{D}}\mathcal{C}\wedge\mathbf{d}\mathcal{C} = \text{const}$ along any Hamiltonian flow on $\mathscr{P}$. ■

We notice that a simple consequence of the above theorem is the

Corollary 5.5 *With the initial condition $\boldsymbol{\mu}_0\cdot\mathbf{n}_0 = 0$, the circulation theorem for the equations of liquid crystals reads as*

$$\frac{d}{dt}\oint_{\gamma(t)}\left(m + \boldsymbol{\mu}\cdot(\mathbf{n}\times\nabla\mathbf{n})\right) = 0\,,$$

which follows directly from the property $\left(\partial_t + \pounds_{\delta h/\delta m}\right)\mathcal{C} = -\mathbf{d}p$. Similar arguments on the relation between the Kelvin-Noether circulation and the helicity invariant also hold for superfluid dynamics [20].

Remark 5.6 (Two dimensional flows) It is important to notice that formula (5.13) provides a whole class of Casimir functions for 2D flows. Indeed, it is well known that, in two dimensions, any function $\Phi(\omega)$ of the vorticity $\omega = \hat{\mathbf{z}}\cdot\operatorname{curl}(m)$ yields a Casimir $C = \int\Phi(\omega)$ and thus, by (5.13), any function $\Phi(\omega + \hat{\mathbf{z}}\cdot\operatorname{curl}(\nabla\mathbf{n}\cdot\boldsymbol{\pi}))$ produces a new Casimir

$$C = \int_{\mathcal{D}}\Phi(\omega + \{\pi_a, n^a\}) = \int_{\mathcal{D}}\Phi\big(\omega + \varepsilon_{abc}\{\sigma^b n^c, n^a\}\big)$$

for 2D flows of uniaxial nematics. Here, the operation $\{\cdot,\cdot\}$ is the canonical Poisson bracket in the planar coordinates (x, y). This observation may have important consequences in terms of the stability properties of liquid crystals, following the approach in [23].

5.3 Reduction schemes for uniaxial nematic liquid crystals

Starting from the same Lagrangian L_{n_0} on the tangent bundle of $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, SO(3))$ we have obtained the Ericksen-Leslie fluid equations in two different ways. For completeness, this section considers nonvanishing r.

First, by considering as symmetry group the semidirect product $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3))$, we obtain the equations of motion by Euler-Poincaré reduction, following the general approach described in §2.1. The reduced space is given by $\big(\mathfrak{X}_{\mathrm{vol}}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))\big) \times \mathcal{F}(\mathcal{D}, S^2)$ and the reduction map reads

$$\begin{aligned}(\eta, \dot\eta, \chi, \dot\chi, \mathbf{n}_0) \longmapsto & \big((\eta, \dot\eta, \chi, \dot\chi)(\eta, \chi)^{-1}, (\eta, \chi)\mathbf{n}_0\big) \\ & = \big(\dot\eta \circ \eta^{-1}, (\dot\chi\chi^{-1}) \circ \eta^{-1}, (\chi\mathbf{n}_0) \circ \eta^{-1}\big) \\ & =: (u, \boldsymbol{\nu}, \mathbf{n})\end{aligned}$$

according to the general formula (2.1). This approach is consistent with that given in [13].

Second, we use Lagrangian reduction by stages (see [5]), and obtain the motion equations on the reduced space $\mathfrak{X}_{\mathrm{vol}}(\mathcal{D}) \times T\mathcal{F}(\mathcal{D}, S^2) \times \mathcal{F}(\mathcal{D})$. First we apply Lagrange-Poincaré reduction for the internal structure, with respect to the mechanical connection (here we choose $\mathbf{n}_0 = (0, 0, 1)$). We get

$$\begin{aligned}(\eta, \dot\eta, \chi, \dot\chi, \mathbf{n}_0) \longmapsto & \big(\eta, \dot\eta, \chi\mathbf{n}_0, (\dot\chi\chi^{-1}) \times (\chi\mathbf{n}_0), (\chi\mathbf{n}_0)\cdot\dot\chi\chi^{-1}\big) \\ & =: (\eta, \dot\eta, \bar{\mathbf{n}}, \bar{\boldsymbol{\nu}} \times \bar{\mathbf{n}}, \bar{\boldsymbol{\nu}}\cdot\bar{\mathbf{n}}) =: (\eta, \dot\eta, \bar{\mathbf{n}}, \dot{\bar{\mathbf{n}}}, \bar r).\end{aligned}$$

Then, we use Lagrangian reduction with respect to the fluid variables and have

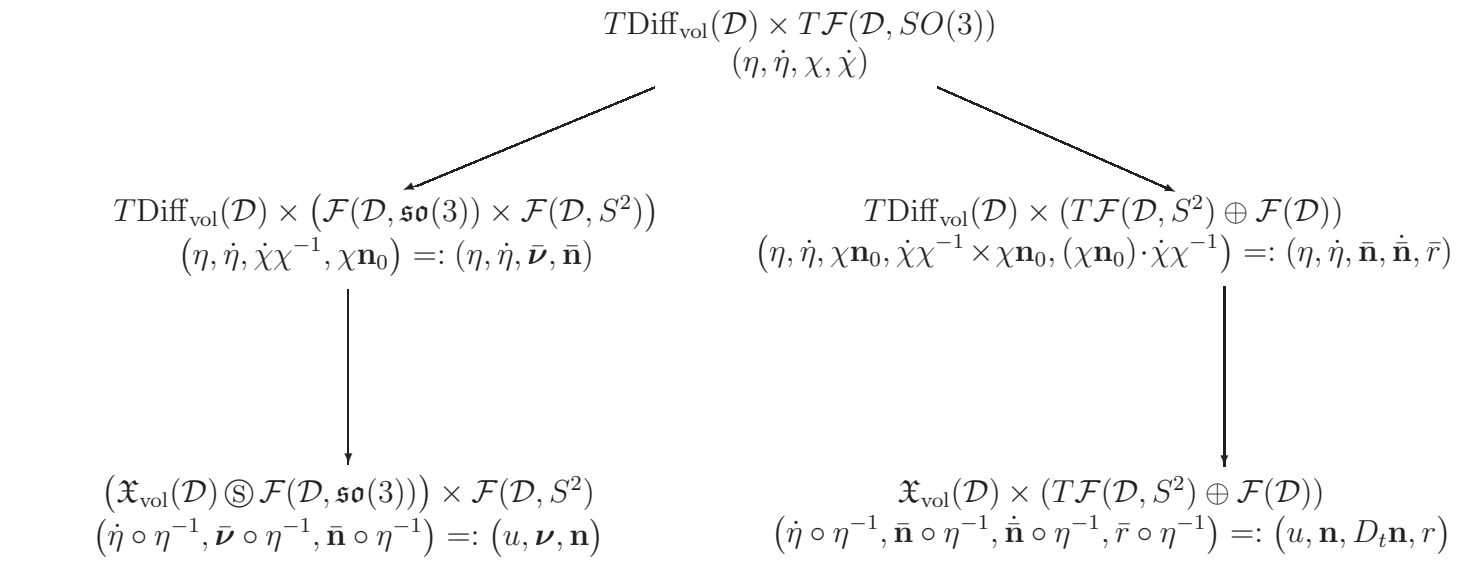
$$(\eta, \dot\eta, \bar{\mathbf{n}}, \dot{\bar{\mathbf{n}}}, \bar r) \longmapsto (\dot\eta \circ \eta^{-1}, \bar{\mathbf{n}} \circ \eta^{-1}, \dot{\bar{\mathbf{n}}} \circ \eta^{-1}, \bar r \circ \eta^{-1}) =: (u, \mathbf{n}, D_t\mathbf{n}, r)\,.$$

This approach is consistent with that given in [15]. Note the following important relation between the intermediate and fully reduced variables. We have

$$\begin{aligned}&\bar{\mathbf{n}} = \chi\mathbf{n}_0, \quad \mathbf{n} = (\chi\mathbf{n}_0) \circ \eta^{-1} = \bar{\mathbf{n}} \circ \eta^{-1} \\ &\bar{\boldsymbol{\nu}} = \dot\chi\chi^{-1}, \quad \boldsymbol{\nu} = (\dot\chi\chi^{-1}) \circ \eta^{-1} = \bar{\boldsymbol{\nu}} \circ \eta^{-1} \\ &\bar r = (\chi\mathbf{n}_0)\cdot\dot\chi\chi^{-1} = \bar{\boldsymbol{\nu}}\cdot\bar{\mathbf{n}}, \quad r = \boldsymbol{\nu}\cdot\mathbf{n} = \bar r \circ \eta^{-1}.\end{aligned}$$

One can pass from one approach to the other by using the same transformations (4.13) and (4.14) as for the single particle case, except that now they are applied to vector valued functions instead of vectors. This clarifies the link between the approaches given in [15] and [13]. The diagram below clarifies the various reduction processes involved in the hydrodynamics of nematic

liquid crystals.



6 Conclusions

This paper has developed symmetry-reduction methods for systems with broken symmetry, whose main example consists of nematic particles in liquid crystals (ordinary or biaxial). After extending the Euler-Poincaré theory, this has been applied to the case of a transitive group action on an order parameter manifold, thereby showing how the latter carries its natural coset structure typically appearing in symmetry breaking. This setting has been applied to nematic particles ($SO(3)/O(2)$), biaxial nematics ($SO(3)/D_2$) and V-shaped molecules ($SO(3)/\mathbb{Z}_2$), and its general validity can be used to reproduce dynamics on any order parameter space. Moreover, the Euler-Poincaré and Lie-Poisson dynamics for nematic and biaxial systems have been formulated also in terms of the alignment tensors, which are widely used in the Landau-de Gennes theory.

As a further step, the Lagrangian reduction technique (Lagrange-Poincaré reduction) has been applied directly to the space $(TG)/G_0$, where $G_0 \subset G$ is the isotropy group determining the "breaking symmetry", which is left in the system when the full G-symmetry is broken. This technique has showed how the reduced configuration space may be identified with the coset G/G_0 (order parameter space), provided the adjoint bundle term in $\widetilde{\mathfrak{g}_0}$ can be set to zero. This is indeed the case for nematic particles, so that the resulting dynamics is given by the Ericksen-Leslie equation on $SO(3)/O(2)$ for a single director. The choice of the mechanical connection enabled us to explain why Ericksen-Leslie dynamics can be considered as ordinary Hamiltonian dynamics on the projective plane, upon setting $\mathbf{n} \cdot \boldsymbol{\nu} = 0$ for consistency with the rod-like nature of nematic particles. Moreover, the mechanical connection produced explicit transformations that relate directly the Euler-Poincaré and the Lagrange-Poincaré approaches. The Lagrange-Poincaré approach restricts to the case when the isotropy subgroup $G_0 \subset G$ is a Lie group with $\dim G_0 \geq 1$, which excludes the case when G_0 is discrete. It is then an interesting open question whether it is possible to extend the Lagrange-Poincaré reduction process $(TG)/G_0$ to account for a discrete symmetry group G_0.

All the above techniques have been extended at the continuum level, to produce also hydrodynamic models, thereby recovering the well known Ericksen-Leslie equation for the dynamics

of the director field. As a result of this extension, all the geometric features of the microscopic particle dynamics are naturally transfered at the macroscopic fluid level without substantial modifications. Interestingly enough, the application of Lagrangian reduction to the hydrodynamics of nematic liquid crystals produces a "generalized Lie-Poisson" Hamiltonian structure on $\mathfrak{g}^* \times P$, involving a Lie algebra $\mathfrak{g}$ and a Poisson manifold P. This type of Hamiltonian structure was discovered in [26] and it appears very frequently in hydrodynamic physical models as well as in other contexts, such as imaging sciences. The last part of this paper applied the theory on this generalized Lie-Poisson construction to produce new Poisson bracket structures, as well as new Casimir functions for liquid crystal dynamics. In particular, two explicit expressions for the helicity of uniaxial nematic liquid crystals have been presented in (5.14) and (5.15).

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